The effect of compressibility on the mixing of Lagrangian tracers is analyzed in chaotic stirred flows. Mixing is studied in terms of the finite-time Lyapunov exponents. Mixing and clustering of passive tracers surrounded by Lagrangian coherent structures is observed to increase with compressibility intensity. The role of the stirring rate and compressibility on mixing and clustering has been analyzed.

DOI: 10.1103/PhysRevE.89.022917  
PACS number(s): 89.75.Kd, 47.70.Fw, 47.70.Pq

I. INTRODUCTION

The importance of Lagrangian analysis to understand complex transport problems in fluids has been established during the last decade (see [1–3] and references therein for a review). Among other techniques, finite-time Lyapunov exponents (FTLE) are used extensively to quantify mixing and especially to extract persisting transport patterns in the flow, the so-called Lagrangian coherent structures (LCS) [4,5]. The propagation of passive and active tracers in chaotic flows is a field of study of enormous interest for a variety of different disciplines, ranging from biology to chemistry and physics and as far as financial mathematics and social studies [6]. Many studies have investigated stirring effects on nonlinear incompressible flows, however, this effect has not attracted very much attention in compressible flows, or the compressibility effects have been neglected, as, for example, in atmospheric flows. Compressible flows are physically relevant not only at large Mach numbers, but they are a relevant issue in combustion [7], in front propagation in reactive media [8], or in plankton dynamics in turbulent flows [9]. On the other hand, the motion of passive tracers in compressible turbulent flows has attracted some attention recently, both theoretically and experimentally [10].

The transport of inertial (finite-size) particles in fluids shows properties typical of compressible fluids, even in incompressible flows. The most unexpected behavior is the formation of clusters of particles out of an initially homogeneous distribution [11,12]. The clustering of inertial particles has many natural and industrial applications such as rain generation [13], point-source pollutant distribution [14], composite materials [15], advection of particles floating on the surface of an incompressible fluid [16], aggregation of radiosonde balloons in the atmosphere [17], or the formation of planetesimals in the early Solar system [18]. In the limit of vanishing Stokes drag, inertial particles recover the motion of Lagrangian tracers and no clustering should be expected [12]. However, passive tracers moving on the surface of an incompressible flow may lead to the formation of cluster structures [11].

In this paper, we address the process of Lagrangian tracers’ clustering subject to a chaotic stirred compressible flow. The combined effect of compressibility intensity and stirring rate leads to the formation of clusters which are characterized in terms of the FTLE field.

II. MODELS

We investigate the effects of compressibility in two-dimensional chaotic flows; the chaotic shear flow and the periodically varying double-gyre flow.

In the first case, the velocity field is modeled by a two-dimensional time-periodic flow capable of producing repeated stretching and folding of fluid parcels, a common characteristic of chaotic mixing. Thus, the velocity field \( (u, v) \) consists of a periodic shear flow [2,19] given by

\[
(u, v) = \frac{\rho_0}{\rho} [A \sin(2\pi y/L + \phi_n), 0],
\]

\[
nT \leq t < (n + 1/2)T,
\]

\[
(u, v) = \frac{\rho_0}{\rho} [0.A \sin(2\pi x/L + \phi_n)],
\]

\[
(n + 1/2)T \leq t < (n + 1)T,
\]

over the domain \([-20,20] \times [-20,20]\), \(L = 40\), \(T = 1/\nu\) and \(A\) are the period and shear amplitude of the flow, respectively, \(n\) is the period number, and \(t\) is the time [20].

On the other hand, the periodically varying double-gyre flow [4,21] is used as a standard test case for LCS and can be considered a local view of a gulf stream ocean front. In this case, the flow is described by the velocity field

\[
u = -\pi A \frac{\rho_0}{\rho} \sin[\pi f(x,t)] \cos(\pi y),
\]

\[
v = \pi A \frac{\rho_0}{\rho} \frac{\partial f}{\partial x} \cos[\pi f(x,t)] \sin(\pi y),
\]

where

\[
f(x,t) = a(t)x^2 + b(t)x,
\]

\[
a(t) = u_0 \sin \omega t,
\]

\[
b(t) = 1 - 2u_0 \sin \omega t,
\]

over the domain \([0,2] \times [0,1]\). As in the previous model, \(T = 1/\nu = 2\pi/\omega\) and \(A\) are the period and amplitude of the flow, respectively. For \(u_0 = 0\) the system can be thought of as a time-independent two-dimensional (2D) Hamiltonian system. For this case there is a heteroclinic connection of the unstable manifold of the fixed point \((1,1)\) with the stable manifold of the fixed point \((1,0)\). For \(u_0 \neq 0\) the gyres conversely...
expand and contract periodically in the $x$ direction such that the rectangle enclosing the gyres remains invariant. The periodic perturbation leads to mixing between the two gyres.

To satisfy the continuity equation $\partial_t(\rho u) = 0$, the spatial dependence of the flow density can be written as

$$\rho(x, y) = \rho_0[1 + \epsilon \sin(2\pi x/\lambda) \sin(2\pi y/\lambda)],$$

where $\epsilon$ and $\lambda$ are the compressibility of the flow and wavelength, respectively. Any velocity field of the form $\vec{W} = \{u(x, y), v(x, y)\}/\rho(x, y)$, with $\rho(x, y)$ a periodic function, satisfies the continuity equation, and similar results to those presented here are expected.

For the numerical experiments, initially a cluster of $400 \times 400$ (shear flow) and $600 \times 300$ (gyre flow) tracer particles were regularly spaced in the domains. Then, the Lagrangian trajectories of these particles are computed by integrating the equations above using a fourth-order Runge-Kutta scheme and a fixed time step of $\Delta t = 10^{-3}$. To characterize the coherent Lagrangian transport of the flows, the FTLE $\sigma$ are computed from the trajectories of Lagrangian tracers in the flow [5] as

$$\sigma(x_0, t_0, \tau) = \frac{1}{\tau} \log \sqrt[\tau]{\mu_1[C(x_0)]},$$

where $\mu_1$ is the largest eigenvalue of the right Cauchy-Green deformation tensor of the flow. For the computation of FTLE fields the integration time $\tau$ must be predefined. Basically, the time $\tau$ has to be long enough to allow trajectories to explore the Lagrangian coherent structures present in the flow. The FTLE at a given location $x$ measures the maximum stretching rate over the interval $\tau = t - t_0$ of trajectories starting near the point $x$ at time $t_0$ [4]. Ridges in the FTLE field are used to estimate finite time invariant manifolds in the flow that separates dynamically different regions.

III. RESULTS

We analyze the role of wavelength $\lambda$, at constant compressibility $\epsilon$, on the FTLE fields for the shear flow model in Fig. 1. For small $\lambda$ values, LCS are strongly affected by the density field (3) and the formation of clusters with a grid configuration is clearly visible. As time evolves, this configuration remains approximately constant. For larger $\lambda$ values, the grid-type configuration blurs and LCS and particles evolve freely. For the double-gyre flow, Fig. 2, compressibility perturbation wrinkles the LCS in the small-wavelength limit, whereas for the large wavelengths LCS are slightly distorted since the entire domain is embedded in nearly a wavelength. For both models, LCS distortion is more significant at smaller values of $\epsilon$ for smaller values of $\lambda$.

Due to the specific characteristics of the flows, Lagrangian particles behave differently in both flows. The chaotic shear flow is a combination of two orthogonal motions, each with sinusoidal velocity and density profiles. Each motion acts alternatively for one-half of the flow period favoring the clustering of the Lagrangian particles in the interstices of a net with wavelength $\lambda$. The presence of the random phase $\psi$ in the shear flow allows particles to percolate through the LCS. For the double-gyre flow, the density profile favors some initial clustering of the particles that is broken by the periodically contracting and expanding of the gyres. Then particles move within the interstices between LCS. Figure 3 shows these clusters for both flow models as regions with a compact concentration of particles. The particles belonging to these clusters are characterized by a negative value of $\sigma$. Remember that for an incompressible flow ($\epsilon = 0$) the FTLE values are always positive for both flows.
The temporal behavior of the FTLE is shown in Fig. 4 in terms of the probability distribution. The probability distribution functions (PDFs) for both flows are respectively plotted in Figs. 4(a) and 4(c) for several integration times \( \tau \). Note that as time integration increases, the \( \sigma \) distribution becomes narrower and they approach each other for large \( \tau \). The peak and mean of the distributions are always positive, although a significant portion of the PDFs correspond to negative values of \( \sigma \). These values characterize particles that cluster together. As time increases, the number of particles with \( \sigma < 0 \) decreases with time towards a nonzero value, but the decaying rate is smaller for large compressibility values. For large \( \tau \), the probability of the FTLE obeys the form [22]

\[
P(\sigma) \sim \exp[-\tau S(\sigma)],
\]

where \( S(\sigma) \) is the Cramér's function or the entropy function, and \( S(\bar{\sigma}) = 0 \). In the simplest case, for a Gaussian distribution, the Cramér's function is a parabola, \( S(\sigma) \sim (\sigma - \bar{\sigma})^2 \).

Finally, the effects of the flow amplitude \( A \) and stirring period \( T \) were studied in both models. For large values of \( A \), mixing is mainly characterized by the chaotic flow, clustering diminishes, and correspondingly, the percentage of particles with \( \sigma < 0 \) decreases.

On the other hand, the clustering behavior of the Lagrangian particles on the stirring rate depends on the characteristics of the flow model as it is shown in Fig. 6. For \( \nu \rightarrow 0 \), \( T \rightarrow \infty \), periodic forcing disappear and the flows become time independent. For the double-gyre flow, gyres neither expand nor contract, and some particles move within the interstices of the LCS and aggregate in those areas with large compression \( \rho/\rho_0 > 1 \). However, for the chaotic shear flow, the periodic
FIG. 5. (Color online) Dependence of the mean value of the (a) FTLE $\bar{\sigma}$, (b) standard deviation, (c) percentage of Lagrangian tracers with $\sigma < 0$, (d) number of clusters, and (e) clusters’ size on the compressibility $\epsilon$ for the double-gyre flow. Clusters were estimated by using the subtractive clustering method implemented through the MATLAB function SUBCLUST [24]. Cluster size is a measure of the mean range of influence of the cluster centers. Set of parameters as in Fig. 2. $\lambda = 0.2$ (circles), $\lambda = 0.6$ (squares), and $\lambda = 1.0$ (rhombi).

Thus, for the chaotic shear flow, LCS follow the maxima of the density profile, and particles become less and less influenced by the velocity field and remain enclosed by the LCS, not allowing the formation of new clusters with time. Between both limits, the number of clusters and particles with $\sigma < 0$ increases with $\nu$ and saturates at larger values of the stirring rate for any $\epsilon$ and $\lambda$. For the double-gyre flow, as $\nu \to \infty$ its dynamics is strictly the same as for $\nu \to 0$ and some clustering is expected. For intermediate stirring rates, the motion of fluid particles is favored and more particles can be attracted to areas with large compression.

IV. CONCLUSION

We have studied the effects of compressibility in two chaotic stirred flows. The compressibility is controlled by the parameter $\epsilon$. Two velocity fields were considered: a chaotic shear flow and a periodic double-gyre flow. In the considered flows, mixing is strongly affected by compressibility, and the compressibility field forces a strong localization of density [8,11]. The formation of clusters separated by Lagrangian coherent structures has been analyzed in terms of $\epsilon$. Clustering has been defined as the formation of patches of particles with negative Lyapunov exponent ($\sigma < 0$). The aggregation of particles on these chaotic compressible flows is not a transient phenomenon. The Cramér’s function is observed to collapse to a single curve as time goes on (Fig. 4), while the PDF of the FTLE attains nonzero values for $\sigma < 0$. 

FIG. 6. (Color online) Dependence of the mean value of the FTLE $\bar{\sigma}$ and percentage of Lagrangian tracers with $\sigma < 0$ on the mixing rate $\nu$, for (a,b) the chaotic shear flow and (c,d) double-gyre flow. Set of parameters as in Fig. 3.
Cluster formation is enhanced as compressibility increases (Fig. 5) based on the combination of particles attracted to areas with large compression $\rho/\rho_0 > 1$, and the detaching of patches of particles from these initial clusters that wander among the chaotic flow. The effects of the density grid correlation length $\lambda$ has also been analyzed. Mixing and clusters’ size and number diminish with increasing $\lambda$, but these clusters become more populated. Depending on the flow dynamics, clustering is observed to occur only for the double-gyre flow for small values of the mixing rate (Fig. 6), while for large-enough stirring rates the flow is quenched and clusters survive forever for any wavelength and compressibility intensity.

Geophysical flows such as ocean and atmospheric fields are far from being homogeneous and isotropic, as large-scale eddies determine their dynamics. Thus, our findings should be relevant to the motion of passive tracers as drifting buoys or pollutants, or to the distribution of active chemicals, in these nonhomogeneous flows.

**ACKNOWLEDGMENTS**

This work was supported by Xunta de Galicia under Research Grant No. CN2012/315. The computational part of this work was done in the Supercomputing Center of Galicia, CESGA. The author gratefully acknowledges the helpful comments of Dr. F. Huhn and two anonymous reviewers.