Fractal Diffraction Properties in Tapered GRIN Media

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ABSTRACT

This paper analyzes the light propagation in a tapered gradient-index medium when the input signal is a binary fractal function. Analytical expressions for the di racted eld at the Fourier transform plans of the inhomogeneous medium are obtained. The results show self-similarity properties at speci c plans of the GRIN.

Keywords: Di raction and gratings, Imaging systems

1. INTRODUCTION

The interesting geometry of the fractals and their relationship with nature have been introduced in the sciences by Mandelbrot.[1] This formalism has been applied in di erent elds.[2] Di raction by fractal apertures has been studied by several authors in homogeneous media.[3–6]

Di raction by these structures has been analyzed in a quadratic refractive index medium.[7] The authors use the relationship between the propagation in such inhomogeneous media and the FRT (fractional Fourier transform).[8]

In the present paper we study the di raction by fractals in more general inhomogeneous media like the tapered gradient-index media.[9] Di raction by di erent kind of binary apertures has been analyzed in tapered GRIN media, in particular the self-image phenomenon has been studied.[10,11]

In this paper we analyze the evolution of the propagated eld with uniform illumination when a binary fractal is located at the entrance of a tapered GRIN medium. The plans where the GRIN medium generates the Fourier transform of the aperture are studied particularly. Analytical expressions for the eld are obtained.

2. PROPAGATION OF A CANTOR BAR IN A TAPERED GRIN MEDIA

The tapered GRIN media are characterized by a transverse parabolic refractive index modulated by an axial index, with the form[9]:

$$n^2(x,z) = n_0^2[1 \quad g^2(z)x^2], \tag{1}$$

where n_0 is the index at the z optical axis and $g(z)$ is the taper function that describes the evolution of the transverse index along z axis.

We will study the light propagation in a tapered gradient-index medium considering that a Cantor bar is located at the entrance like a binary di ractive grating. The Cantor bars are binary functions $C_N(x)$ with domain $x \in [\quad ; \quad]$ and N positive integer. For $N = 1$ the function takes de value 1 in $x \in [\quad ; \quad /3]$ and $[\ /3; \ ]$, and it takes de value 0 in $x \in (\quad /3; \ /3)$. The next Cantor level repeats the same triadic structure at each interval of value 1, the process is repeated in the same way for the next levels in each interval of value 1. In general at level N the function $C_N(x)$ has 2^N intervals with value 1, and takes the value 0 in the remaining domain. Therefore a grating with transmittance function represented by $C_N(x)$ will have 2^N open slits of width $2\ /3^N$ fractally distributed. We call x_j, x_j^- and x_j^+ the middle, beginning and end points of the $j \quad th$ slit, where the slits are ordered from $x = \quad$ to $x = \ $, with j integer and $1 \quad j \quad 2^N$.

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The entrance o the medium, then the transmittance function is given by

$$T(x_0) = C_N(x_0), \tag{2}$$

where $C_N(x_0)$ is the Cantor bar of level $N \geq 0$.

When the hybrid optical structure formed by this one-dimensional object and the tapered GRIN medium is illuminated by a coherent nonuniform beam, the complex amplitude distribution at $z = 0$ will be

$$\phi(x_0) = T(x_0) \quad _0(x_0), \tag{3}$$

where

$$_0(x_0) = \frac{w_0}{w(0)} ^{1/2} \exp\left[i\varphi(0)\right] \quad [x_0; U(0)] \tag{4}$$

is the complex amplitude distribution due to a Gaussian illumination of wavelength λ, and

$$[x_0; U(0)] = \exp\left(i\frac{\pi U(0)x_0^2}{\lambda}\right) \tag{5}$$

is the quadratic phase factor of the cylindrical Gaussian beam. The beam parameters at distance Z_0 from the waist plane of diameter $2w_0$ are given by the complex curvature U and the on-axis phase φ, that is

$$U(0) = \frac{1}{R(0)} + i\frac{\lambda}{\pi w^2(0)}, \tag{6}$$

$$\varphi(0) = \tan^{-1}\left(\frac{\lambda Z_0}{\pi w_0^2}\right), \tag{7}$$

where $R(0)$ and $w(0)$ are the radius of curvature and the half-width at $z = 0$, respectively.

The eld distribution in the tapered GRIN medium at $z > 0$ is given by the integral equation[9]

$$\phi(x; z) = \int_{-\infty}^{+\infty} \phi(x_0) K(x, x_0; z) \mathrm{d}x_0, \tag{8}$$

where K is the one-dimensional optical propagator of this medium expressed as

$$\begin{aligned} K(x, x_0; z) \quad = \quad & \frac{n_0}{i\lambda H_1(z)} ^{1/2} \exp\left(i\frac{2\pi}{\lambda}n_0 z\right) \\ & \times \exp \quad i\frac{\pi n_0}{\lambda H_1(z)}\left[x^2 \dot{H}_1(z) + x_0^2 H_2(z) \quad 2xx_0\right]\Big\}, \end{aligned} \tag{9}$$

where H_1, H_2 and $\dot{H}_1$, $\dot{H}_2$ are the position and the slope of the axial and eld rays at z, respectively, and dot being the derivative with respect to z.

Insertion of Eq. (2) into Eq. (8) gives

$$\phi(x; z) \propto \int_{-\xi}^{+\xi} C_N(x_0) \exp \quad i\frac{\pi}{\lambda}\left(U(0) + \frac{n_0 H_2(z)}{H_1(z)}\right) x_0^2 \quad \frac{2\pi i}{\lambda}\frac{n_0 x}{H_1(z)} x_0 \quad \mathrm{d}x_0 = \tag{10}$$

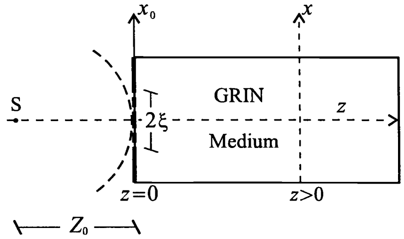

Figure 1. Geometry of a tapered GRIN medium with a Cantor aperture illuminated by a uniform cylindrical beam.

$$= \sum_{j=1}^{2^N} \int_{x_j^-}^{x_j^+} \exp \; i\frac{\pi}{\lambda}\left(U(0) + \frac{n_0 H_2(z)}{H_1(z)}\right) x_0^2 \quad \frac{2\pi i}{\lambda}\frac{n_0 x}{H_1(z)} x_0 \;\; \mathrm{d}x_0, \tag{11}$$

where slit positions have been employed.

Supposing uniform cylindrical illumination we have $R(0) \to Z_0$, $w(0) \to \infty$, and $z_R \to \infty$, where Z_0 is the curvature radius of the incident beam at the entrance (see Figure 1). Under these conditions the expressions (10) and (11) are reduced to

$$\phi(x;z) \propto \int_{-\xi}^{+\xi} C_N(x_0) \exp \; i\frac{\pi}{\lambda}\left(\frac{1}{Z_0} + \frac{n_0 H_2(z)}{H_1(z)}\right) x_0^2 \quad \frac{2\pi i}{\lambda}\frac{n_0 x}{H_1(z)} x_0 \;\; \mathrm{d}x_0 = \tag{12}$$

$$= \sum_{j=1}^{2^N} \int_{x_j^-}^{x_j^+} \exp \; i\frac{\pi}{\lambda}\left(\frac{1}{Z_0} + \frac{n_0 H_2(z)}{H_1(z)}\right) x_0^2 \quad \frac{2\pi i}{\lambda}\frac{n_0 x}{H_1(z)} x_0 \;\; \mathrm{d}x_0. \tag{13}$$

3. FOURIER TRANSFORM CONDITIONS

The di racted eld reproduces the Fourier transform of the transmittance function under the next conditions

$$H_2(z) = \quad \frac{H_1(z)}{n_0 Z_0}, \tag{14}$$

then the expression of Eq. (13) reduces to

$$\phi(x;z) \propto \sum_{j=1}^{2^N} \int_{x_j^-}^{x_j^+} \exp[\quad i\mu x x_0]\, \mathrm{d}x_0, \tag{15}$$

where

$$\mu = \frac{2\pi}{\lambda}\frac{n_0}{H_1(z)} \; . \tag{16}$$

The integral of Eq. (15) can be expressed by

$$\phi(x;z) \propto \quad \frac{1}{i\mu x}\sum_{j=1}^{2^N} \exp \qquad i\mu x x_0 \Big|_{x_j^-}^{x_j^+} \quad = \quad \frac{1}{i\mu x}\sum_{j=1}^{2^N} \; \exp(\quad i\mu x x_j^+) \quad \exp(\quad i\mu x x_j^-) \; . \tag{17}$$

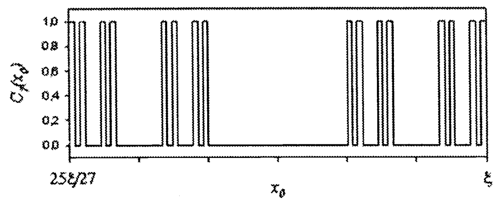

Figure 2. Part of the function Cantor bar with $N = 7$.

Using that $x_j^+ = x_j + \frac{\xi}{3^N}$ and $x_j^- = x_j \quad \frac{\xi}{3^N}$ we obtain

$$\phi(x;z) \propto \quad \frac{1}{i\mu x}\sum_{j=1}^{2^N} \exp\left(\quad i\mu x_j x\right) \;\; \exp\left(\quad i\mu \frac{\quad}{3^N} x\right) \quad \exp\left(i\mu \frac{\quad}{3^N} x\right) \tag{18}$$

$$= 2\frac{\sin(\mu \frac{\xi}{3^N} x)}{\mu x}\sum_{j=1}^{2^N} \exp\left(\quad i\mu x_j x\right), \tag{19}$$

the rst factor in Eq. (19) represents the Fraunhofer di raction amplitude of a single slit of width $\frac{2\xi}{3^N}$, and the second factor represents the interference of the 2^N slits.

Using that

$$\sum_{j=1}^{2^N} \exp\left(\quad i\mu x_j x\right) = \prod_{m=1}^{N} 2\cos\left(\frac{2\mu \;\, x}{3^m}\right) = 2^N \prod_{m=1}^{N} \cos\left(\frac{2\mu \;\, x}{3^m}\right) \tag{20}$$

then $\phi(x;z)$ can be expressed by

$$\phi(x;z) \propto 2 \;\, \left(\frac{2}{3}\right)^N S_N(x) F_N(x) \tag{21}$$

where

$$S_N(x) = \frac{\sin(\mu \frac{\xi}{3^N} x)}{\mu \frac{\xi}{3^N} x} = \mathrm{sinc}\left(\mu \frac{\quad}{3^N \pi} x\right), \tag{22}$$

and

$$F_N(x) = \prod_{m=1}^{N} \cos\left(\frac{2\mu \;\, x}{3^m}\right) = \prod_{m=0}^{N-1} \cos\left(\frac{3^m 2\mu \;\, x}{3^N}\right). \tag{23}$$

The function $F_N(x)$ contains the self-similar property of the di racted eld (17), it has a recursive form:

$$F_{N+1}(x) = \cos\left(\frac{2\mu \;\, x}{3}\right) F_N\left(\frac{x}{3}\right), \tag{24}$$

in the limit $N \to \infty$ we have

$$F_\infty(x) = \cos\left(\frac{2\mu \;\, x}{3}\right) F_\infty\left(\frac{x}{3}\right), \tag{25}$$

the expression in Eq. (25) shows that when $N \to \infty$, $F_N(x)$ and $\phi(x;z)$ are self-similar with respect to the scale transformation by factor $\varepsilon = 1/3$, the same scale factor of $C_N(x)$.

The axial localization of the Fourier plans can be obtained if we take into account that the axial and eld rays are given by[9]

$$H_1(z) = [g_0 g(z)]^{-1/2} \sin\left[\int_0^z g(z')\mathrm{d}z'\right], \tag{26}$$

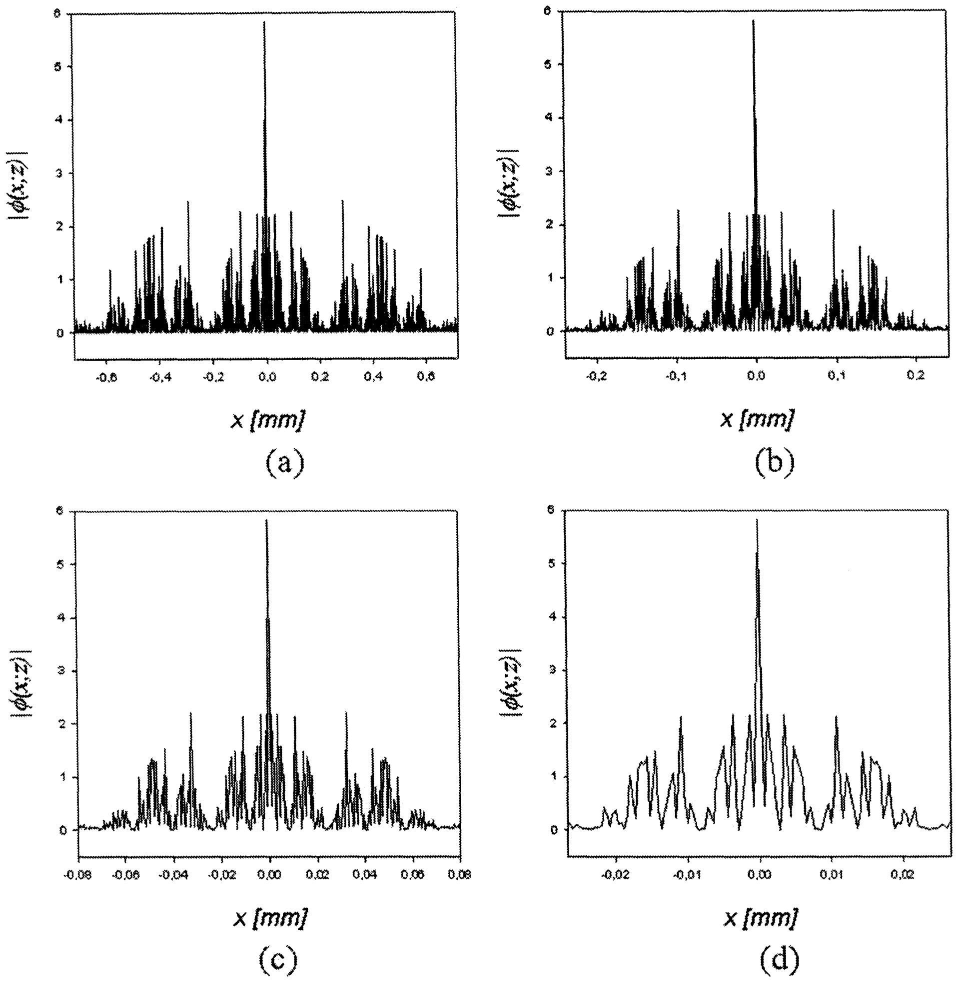

Figure 3. Di racted field at the first Fourier plan in the tapered GRIN medium.

$$H_2(z) = \frac{g_0}{g(z)}^{1/2} \cos\left[\int_0^z g(z')\mathrm{d}z'\right], \tag{27}$$

where g_0 is the value of $g(z)$ at $z = 0$. Substituting these expression into the condition (14) the Fourier distance can be calculated.

We consider a tapered GRIN medium with a divergent linear taper function given by

$$g(z) = \frac{g_0}{1 + z/L}, \tag{28}$$

where L is the distance from $z = 0$ to the common apex of the equi-index lines of the refractive-index pro le.[9] In this medium we suppose that the Cantor $C_7(x)$ is located at the entrance like a di ractive grating. In Figure 2 we show part of the Cantor $C_7(x)$, we introduce $C_7(x)$ in Eq. (12) and calculate numerically the integral at distance $z = z_F$ corresponding to the rst Fourier transform plan of the GRIN, where $g(z)$ is given by (28), then we take the modulus of $\phi(x; z_F)$. We have considered $n_0 = 1.5$, $g_0 = 0.01mm^{-1}$, $\lambda = 0.7\mu m$, $L = 1mm$, $= 1mm$, and $Z_0 = 15mm$.

The results of the calculus of the expression (12) can be seen in Figure 3(a). In Figure 3(b) the same eld is magni ed a factor 3, in Figure 3(c) a factor 3^2, and in Figure 3(d) a factor 3^3. These graphics show that $|\phi(x; z_F)|$ results a fractal with respect to the scale transformation by the factor 3, like the function $C_7(x)$. Therefore the result and the analysis of Eq. (25) are veri ed.

4. CONCLUSIONS

We have considered the di raction when a triadic Cantor bar is located like a di ractive grating at the entrance of a tapered gradient-index medium. In the plans where the medium produce the Fourier transform of the transmittance aperture the amplitude distribution results self-similar and with the same scaling property of the di raction grating. We have obtained analytical expression for the eld in such plans, and computer simulations that corroborate the analysis.

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