

Walsh–Hadamard analysis applied to the study of light propagation in a tapered gradient-index medium

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The analysis of light propagation in a tapered gradient-index (GRIN) medium when the input signal is a binary function is considered with the Walsh–Hadamard analysis. The study of the evolution of the sequency spectrum through the GRIN medium by the Walsh–Hadamard transform confirms the self-imaging effect.

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1. INTRODUCTION

The orthogonal set of complex exponential functions plays a fundamental role in the study of linear systems. In fact, the Fourier analysis, based on such functions, is also one of the most important mathematical tools in physics and other fields. In optics the inherent ability of the lenses to decompose optical signals into their spatial frequencies permitted the development of Fourier optics and several techniques of optical information processing.¹

Another orthogonal and complete set of functions is that constituted by the Walsh functions.² This set defines the Walsh–Hadamard transform, which is the equivalent of the Fourier transform. The Walsh–Hadamard transform establishes the Walsh–Hadamard spectrum or sequency spectrum. The study and processing of the information in such a domain transformation constitute the Walsh–Hadamard analysis, which is as useful as the Fourier analysis.

Walsh functions have been widely employed in the field of communications.^{3,4} In optics the Walsh analysis has been applied to digital image processing.⁵ Moreover, since it has been demonstrated that the binarized Walsh functions verify the Talbot and the Lau effects⁶ under certain conditions,⁷ some applications of these properties have been proposed.^{8,9} These studies show the simplicity of applying the Walsh analysis in the study of certain problems.

In a previous paper¹⁰ we demonstrated that the Walsh functions verify the integer and the fractional self-image phenomena when a Gaussian wave front is propagated into a tapered gradient-index (GRIN) medium. Likewise, the particular definition of the Walsh functions as products of Rademacher functions² has allowed us to demonstrate an interesting relationship between the fractional self-image orders and the different Rademacher functions included in the definition of each Walsh function.

With these results in mind, in the present paper we analyze light propagation through a GRIN medium by

studying the Walsh–Hadamard transform of apertures represented by Walsh functions.

2. WALSH–HADAMARD ANALYSIS

The Walsh functions² form a complete and orthogonal set of functions of rectangular waveforms taking only two amplitude values, +1 and –1. The functions are defined over a limited interval and are characterized by an order number or index N integer. We denote the Walsh functions by

$$\text{Wal}_N(t), \quad t \in [0, \xi], \quad N = 0, 1, 2, \dots \quad (1)$$

One way to obtain the mathematical expression of the Walsh functions is to employ the Rademacher functions:

$$\text{Rad}_r(t), \quad t \in [0, \xi], \quad r = 0, 1, 2, \dots \quad (2)$$

These functions constitute an incomplete and orthogonal set of periodic square waveforms of amplitude +1 and –1 and have a period d_r that decreases with increasing order number or index r :

$$d_r = \xi 2^{1-r}. \quad (3)$$

The Rademacher functions can be derived from sinusoidal functions that have identical zero-crossing positions. In the interval $t \in [0; \xi]$ they can be expressed by¹¹

$$\text{Rad}_r(t) = \text{sign} \left[\cos \left(2^r \pi \frac{t}{\xi} \right) \right]. \quad (4)$$

For our application it is necessary to consider a definition interval $x_0 \in [-\xi/2; \xi/2]$, and then the translation $x_0 = t - (\xi/2)$ is introduced to preserve the shapes and properties of the Rademacher functions, which take the form

$$\text{Rad}_r(x_0) = \text{sign} \left\{ \cos \left[2^r \pi \left(\frac{x_0}{\xi} + \frac{1}{2} \right) \right] \right\}. \quad (5)$$

The Walsh functions can be defined as products of the Rademacher functions by

$$\text{Wal}_N(x_0) = \prod_{r=p}^m [\text{Rad}_r(x_0)]^{g_r}. \quad (6)$$

The integers m , p , and g_r are defined by the binary expansion of N : $N = \sum_{r=p}^m g_r 2^r$, where g_r are the corresponding bits 0 or 1.

Since Walsh functions constitute a complete set, any arbitrary function $f(x_0)$ can be expressed as

$$f(x_0) = \sum_{M=0}^{\infty} \tilde{w}_M \text{Wal}_M(x_0), \quad (7)$$

where \tilde{w}_M are the coefficients of the expansion and can be obtained by

$$\tilde{w}_M = \frac{1}{\xi} \int_{-\xi/2}^{\xi/2} f(x_0) \text{Wal}_M(x_0) dx_0. \quad (8)$$

The coefficients \tilde{w}_M with $M = 0, 1, 2, \dots$ represent the peak amplitudes of the spectral components of $f(x_0)$. The set of these coefficients forms a series ϖ_N that expresses $f(x_0)$ in the sequency domain. In the same way that the Fourier transform represents the frequency spectrum, the set ϖ_N is the Walsh–Hadamard transform (WHT) and represents the sequency spectrum or Walsh–Hadamard spectrum of $f(x_0)$.

3. STATEMENT OF THE PROBLEM

Studying how the evolution of the diffracted field along an inhomogeneous medium works when a Walsh function represents an aperture can help us to comprehend the behavior of the field for any function aperture, given that all functions can be decomposed into Walsh functions. Moreover, studying this problem by employing the Walsh–Hadamard scheme offers a different approach to diffraction in those media.

The refractive index of a tapered GRIN medium characterized by a transverse parabolic refractive index modulated by an axial index is given by

$$n^2(x, z) = n_0^2 [1 - g^2(z)x^2], \quad (9)$$

where n_0 is the index at the z optical axis and $g(z)$ is the taper function that describes the evolution of the transverse index along the z axis. We have taken a quadratic expression for the transverse optical index because all terms of order higher than the second in x have been neglected, the reason being these terms are aberration terms for light propagation in GRIN media. The effect of aberrations is seen to be the introduction of phase distortions within the passband of the spectrum that could have an effect on the fidelity of the signal and imaging processing since the Walsh–Hadamard transform is a unitary transform commonly used in such processing.

We will consider that a one-dimensional binarized Walsh function of extension ξ represents an input aperture of the GRIN medium. Then its transmittance will be

$$T(x_0) = \frac{1}{2} [1 + \text{Wal}_N(x_0)] = W_N(x_0), \quad (10)$$

where $W_N(x_0)$ is the binary expression of the Walsh function $\text{Wal}_N(x_0)$.

When this hybrid optical structure formed by the one-dimensional object and the tapered GRIN medium is illuminated by a coherent nonuniform beam (Fig. 1), the complex amplitude distribution at $z = 0$ will be

$$\phi(x_0) = T(x_0)\psi_0(x_0), \quad (11)$$

where

$$\psi_0(x_0) = \left[\frac{w_0}{w(0)} \right]^{1/2} \exp[i\varphi(0)]\psi[x_0; U(0)] \quad (12)$$

is the complex amplitude distribution due to a Gaussian illumination of wavelength λ and

$$\psi[x_0; U(0)] = \exp\left[i \frac{\pi U(0)x_0^2}{\lambda} \right] \quad (13)$$

is the quadratic phase factor of the cylindrical Gaussian beam. The beam parameters at distance Z_0 from the waist plane of diameter $2w_0$ are given by the complex curvature U and the on-axis phase φ , that is,

$$U(0) = \frac{1}{R(0)} + i \frac{\lambda}{\pi w_0^2(0)}, \quad (14)$$

$$\varphi(0) = \tan^{-1} \left(\frac{\lambda Z_0}{\pi w_0^2} \right), \quad (15)$$

where $R(0)$ and $w(0)$ are the radius of curvature and the half-width at $z = 0$, respectively.

The complex amplitude distribution in the tapered GRIN medium at $z > 0$ is given by the integral equation¹²

$$\phi(x; z) = \int_{-\xi/2}^{\xi/2} \phi(x_0)K(x, x_0; z)dx_0, \quad (16)$$

where K is the one-dimensional optical propagator of this medium expressed as

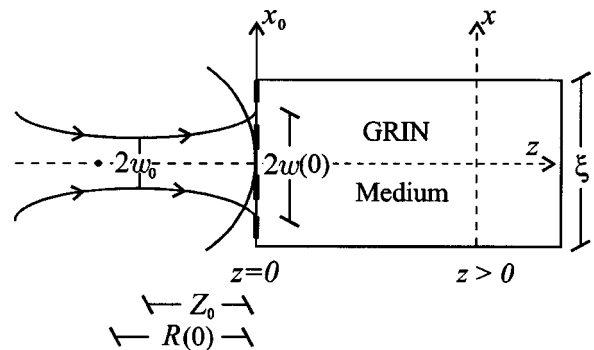


Fig. 1. Geometry of a tapered GRIN medium with a bivalued aperture illuminated by a Gaussian beam.

$$K(x, x_0; z) = \left[\frac{n_0}{i\lambda H_1(z)} \right]^{1/2} \exp\left(i \frac{2\pi}{\lambda} n_0 z \right) \times \exp\left\{ i \frac{\pi n_0}{\lambda H_1(z)} [x^2 \dot{H}_1(z) + x_0^2 H_2(z) - 2xx_0] \right\}, \quad (17)$$

where H_1 , H_2 and \dot{H}_1 , \dot{H}_2 are the position and the slope of the axial and field rays at z , respectively, the dot being the derivative with respect to z .

The axial and field rays are given by¹²

$$H_1(z) = [g_0 g(z)]^{-1/2} \sin\left[\int_0^z g(z') dz' \right], \quad (18a)$$

$$H_2(z) = \left[\frac{g_0}{g(z)} \right]^{1/2} \cos\left[\int_0^z g(z') dz' \right], \quad (18b)$$

where g_0 is the value of $g(z)$ at $z = 0$. They can be expressed as

$$H_1(z) = \frac{u(z)}{\{g_0 g(z)[1 + u^2(z)]\}^{1/2}}, \quad (19a)$$

$$H_2(z) = \left\{ \frac{g_0}{g(z)[1 + u^2(z)]} \right\}^{1/2}, \quad (19b)$$

where

$$u(z) = \tan\left[\int_0^z g(z') dz' \right]. \quad (20)$$

In a previous paper¹⁰ we showed that Walsh functions verify the integer self-image condition at distances z_ν , such that

$$\frac{\text{Re}[G(z_\nu)]}{n_0 |G(z_\nu)|^2} H_1(z_\nu) = \nu \frac{d_p^2}{\lambda}, \quad \nu = 1, 2, \dots, \quad (21)$$

where

$$G(z) = \frac{U(0)H_1(z)}{n_0} + H_2(z), \quad (22)$$

and d_p is the period of $\text{Rad}_p(x_0)$, the Rademacher function with minor index present in the product definition of the Walsh function [see Eqs. (3) and (6)]. Moreover, this minor index p has to verify

$$2^p \gg 1. \quad (23)$$

These results predict that the irradiance will present replicas of the Walsh-function aperture along the medium. Therefore, Walsh–Hadamard spectra of the irradiance should be very simple in such planes, because they should display a single and important maximum in N similar to the Walsh–Hadamard transform of the Walsh function $W_N(x_0)$. Using this, we will study the evolution of the Walsh–Hadamard spectrum along the optical axis.

4. EVOLUTION OF THE IRRADIANCE IN THE TAPERED GRADIENT-INDEX MEDIUM

To study the evolution of the irradiance through the GRIN medium, we assume, for example and without loss of generality, that a binarized form of $\text{Wal}_{896}(x_0) = \text{Rad}_7(x_0)\text{Rad}_8(x_0)\text{Rad}_9(x_0)$ is located at the entrance

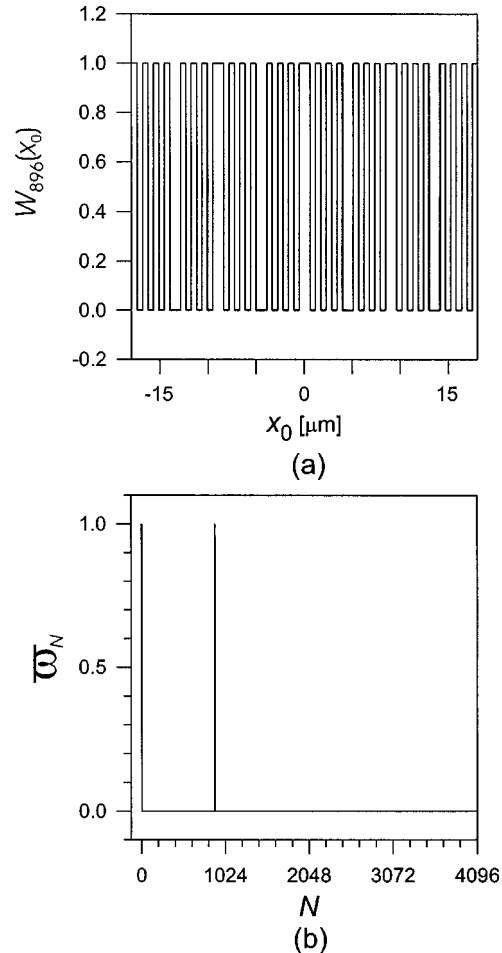


Fig. 2. Input signal: (a) central part of the transmittance function $W_{896}(x_0)$, (b) the WHT.

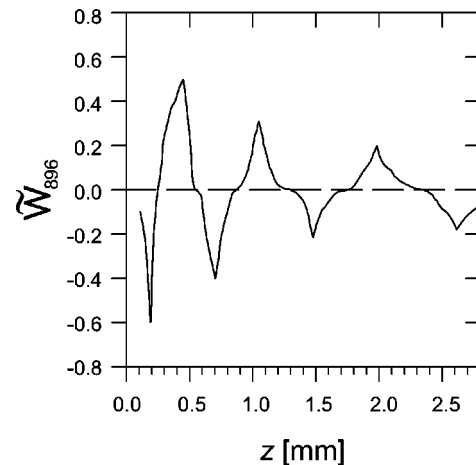


Fig. 3. Evolution of \tilde{w}_{896} in the GRIN medium along the z axis.

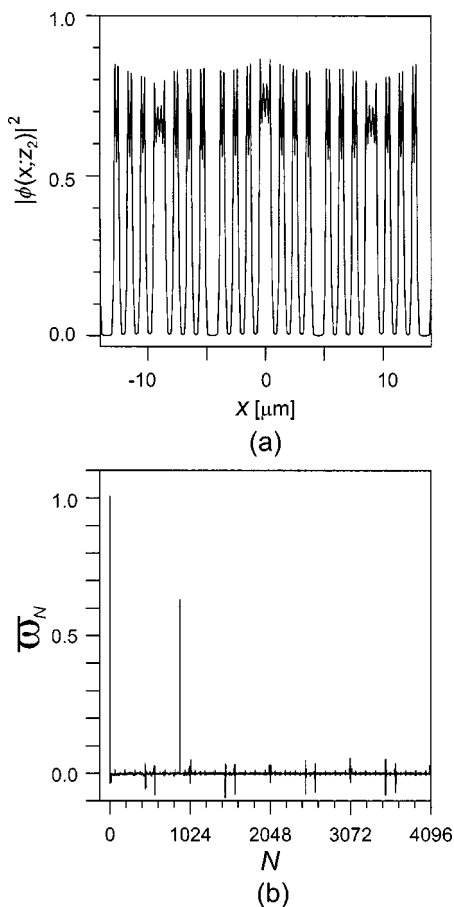


Fig. 4. First positive self-image at $z_2 = 0.42$ mm: (a) central part of the irradiance distribution, (b) the WHT.

of the GRIN medium. The transmittance function of this input signal is given by

$$T(x_0) = \frac{1}{2}[1 + \text{Wal}_{896}(x_0)] = W_{896}(x_0). \quad (24)$$

In this case, we evaluated the irradiance evolution in the GRIN medium from Eq. (16) by computer calculation, assuming uniform illumination as a particular case of nonuniform illumination when we took $R(0) \rightarrow Z_0$, $w(0) \rightarrow \infty$, and $z_R \rightarrow \infty$, with Z_0 being the curvature radius of the uniform wave at the input, and where

$$z_R = \frac{\pi n_0 w^2(0)}{\lambda} \quad (25)$$

is the Rayleigh range that characterizes the Lorentzian profile of Gaussian beam irradiance along the z axis of the GRIN medium.

Moreover, we have considered a tapered GRIN medium with a divergent linear taper function given by

$$g(z) = \frac{g_0}{1 + z/L}, \quad (26)$$

where L is the distance from $z = 0$ to the common apex of the equi-index lines of the refractive-index profile.¹² Calculations were made for $n_0 = 1.5$, $g_0 = 0.01 \text{ mm}^{-1}$, $d_p = 9 \text{ }\mu\text{m}$, $\lambda = 0.7 \text{ }\mu\text{m}$, $L = 1 \text{ mm}$, and $Z_0 = 15 \text{ mm}$.

Figure 2(a) shows the central part of the function aperture $W_{896}(x_0)$, and Fig. 2(b) shows its Walsh-Hadamard

spectrum evaluated by Eq. (8). Such a spectrum was calculated by using a fast transform program and presents a single maximum in $N = 896$; the maximum in $N = 0$ comes from the constant factor in Eq. (24). To study the self-imaging phenomenon in the GRIN medium, we will follow the evolution of \tilde{w}_{896} along the z axis. If the irradiance distribution is a copy of $W_{896}(x_0)$, then its Walsh-Hadamard transform should present a maximum in $N = 896$. Figure 3 displays the evolution of the coefficient \tilde{w}_{896} in a range of z into the GRIN medium. Maximum can be observed at $z = 0.42 \text{ mm}$, $z = 1.05 \text{ mm}$, and z

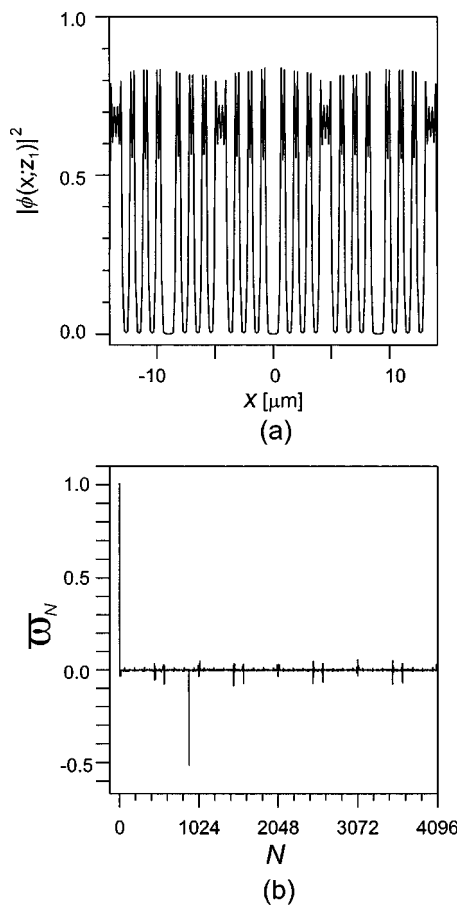


Fig. 5. First negative self-image at $z_1 = 0.19$ mm: (a) central part of the irradiance distribution, (b) the WHT.

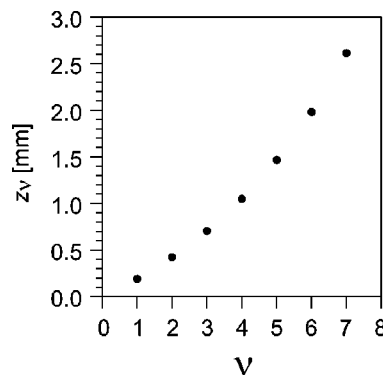


Fig. 6. Self-image distances calculated by Eq. (27) coincide with maximum and minimum of Fig. 3.

= 1.98 mm, and minimum can be seen at $z = 0.19$ mm, $z = 0.70$ mm, $z = 1.47$ mm, and $z = 2.61$ mm; therefore in these positions there are positive and negative self-images, respectively, of $W_{896}(x_0)$. This can be noted in Fig. 4(a), which shows the central part of the irradiance distribution at $z = 0.42$ mm, and can also be observed in Fig. 4(b), its Walsh–Hadamard transform. Figure 4(a) represents the irradiance of the first positive self-image of \bar{w}_{896} into the GRIN medium. In Fig. 4(b) the maximum in $N = 896$ indicates that the irradiance distribution at $z = 0.42$ mm has a strong contribution of $\text{Wal}_{896}(x_0)$, and the maximum in $N = 0$ comes from a constant factor. Another maximum and minimum are depicted in the Walsh–Hadamard transform; they indicate that the remaining energy is distributed in some other orders owing to the diffraction of the field by the finite dimension of the input signal.

Figure 5(a) shows the central part of the irradiance distribution at $z = 0.19$ mm, the first negative self-image of $W_{896}(x_0)$ into the GRIN medium. The Walsh–Hadamard transform of this irradiance is observed in Fig. 5(b), where the minimum at $N = 896$ indicate that the irradiance distribution at $z = 0.19$ mm has a strong contribution of $-\text{Wal}_{896}(x_0)$ and the maximum at $N = 0$ comes from a constant factor.

These self-image distances can also be obtained in a different way by using analytical expressions. The axial localization of Talbot images for a Walsh function $W_N(x_0)$ under these conditions is given by¹⁰

$$z_\nu = L \left\{ \exp \left[\frac{1}{g_0 L} \tan^{-1} \left(\frac{\nu d_p^2 n_0 g_0 Z_0}{Z_0 \lambda - \nu d_p^2} \right) \right] - 1 \right\}. \quad (27)$$

As in Eq. (21), even values of ν correspond to positive self-image distances, and odd values of ν correspond to negative self-image distances. By using this expression we have corroborated the distances obtained above, as shown in Fig. 6.

We have demonstrated that the Walsh–Hadamard analysis is appropriate for studying the diffraction propagation of a single Walsh function through a GRIN medium, since the transform of such a function has a unique characteristic component in the sequency domain. The self-image phenomenon can be easily analyzed with this formalism, because the replicas of the aperture have the same characteristic maximum in the Walsh–Hadamard space. Even in more general situations the advantage of the Walsh–Hadamard analysis can be noted. When the input of the medium is composed of a limited number of Walsh functions, its sequency spectrum will occupy a finite section of the transformed domain. In such cases the Talbot effect can be studied by analyzing a finite number of maximum in the Walsh–Hadamard space. On the other hand, these functions with finite spatial dimension will occupy an infinite section in the Fourier spectrum domain. This has relevance in signal reconstruction for a

given accuracy. For the same reason the use of the Walsh–Hadamard analysis is also recommended when a time-limited signal composed of a superposition of Walsh functions is the input of the GRIN medium.

5. CONCLUSIONS

The Walsh–Hadamard analysis has been employed for studying light propagation through a tapered gradient-index medium with input signal given by a binary Walsh function. The Walsh–Hadamard transform at self-images depends strongly on the value of the index N of the input signal (Walsh function). We have also shown that the analysis of the Walsh–Hadamard spectrum provides the position of positive and negative self-images.

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