Resolving power of a hybrid zone-plate/gradient-index lens system

José Manuel Rivas-Moscoso, Carlos Gómez-Reino*
GRIN Optics Group, Departamento de Física Aplicada, E. U. de Óptica e Optometría
Campus Sur S/N, Universidade de Santiago de Compostela, E15782 Santiago de Compostela, Spain

ABSTRACT

Following the Rayleigh criterion for resolution, we analyze the resolving power of a hybrid optical structure composed of a zone plate and a gradient-index lens at focal planes inside and outside the lens in terms of the type and number of zones of the zone plate. The Fraunhofer approximation can be invoked to study the irradiance at the Fourier planes inside the lens. In all other cases, the Fresnel diffraction integral must be used, although the resolution defined by the Rayleigh criterion still holds. We compare the limit of resolution for the hybrid structure with the resolution of a lens with the same aperture diameter as a function of the number of zones, stating that both coincide for a large number of zones. Finally, we present irradiance profiles with both approximations at various planes inside the lens and compare the results with those obtained by the use of zone plates in free space.

Keywords: gradient-index lenses, diffractive optics

1. INTRODUCTION

The zone plate (ZP) is a classic topic in optics and has attracted widespread attention.[1] A Fresnel ZP is a diffractive device consisting of a series of concentric ring-shaped zones of radii $h_j = \sqrt{j}h_1$, where j assumes an integer value, and period $p = 2h_1^2$, h_1^2 being the square width of each individual zone, which alternately transmit and absorb radiation or else shift the phase of the contribution of every second zone to the total amplitude at an observation point by π rad. An analogy with the conventional lens law can be made; thereby, for illumination of wavelength λ, the plate acts as a multifocal lens with foci at distances $f_m = p/(2m\lambda)$, where m now assumes an odd number.

A hybrid diffractive/gradient-index (GRIN) element composed of a ZP and a GRIN lens with a transverse parabolic refractive index profile given by

$$n^2(r,z) = n_0^2\left[1 - g^2(z)r^2\right], \tag{1}$$

where $r^2 = x^2 + y^2$, n_0 is the index along the optical axis and $g(z)$ is the taper function, has recently been theoretically and experimentally investigated.[2-3] In these investigations, the evolution of the ZP diffraction orders inside a GRIN lens was determined —the foci being, for uniform illumination of radius of curvature d, at the positions f_m resulting from

$$\int_0^{f_m} g(z')\,\mathrm{d}z' = \tan^{-1}\left(\frac{n_0 g_0 p d}{2m\lambda d - p}\right), \tag{2}$$

where $g_0 = g(0)$— and the multifocusing properties of the hybrid structure were demonstrated.

The aim of this paper is to analyze the resolving power of this hybrid element. To determine the limit of resolution, it is necessary to know the irradiance distribution of the image of a point source and then to apply Rayleigh's criterion. The irradiance distribution of the image of a point source can be obtained by calculating the irradiance distribution centered at the principal focal point of the ZP when a plane wave of monochromatic light impinges normally upon the ZP.[4] To tackle this task, we will consider positive (transparent center) and negative (opaque center) Fresnel zone plates of the amplitude and phase and a tapered GRIN lens.

*facgrc@usc.es; phone/fax +34 981 52 19 84; www.usc.es/grinteam

5th Iberoamerican Meeting on Optics and 8th Latin American Meeting on Optics, Lasers, and Their Applications, edited by A. Marcano O., J. L.Paz, Proc. of SPIE Vol. 5622
(SPIE, Bellingham, WA, 2004) · 0277-786X/04/$15 · doi: 10.1117/12.592178

2. FOCAL POSITIONS INSIDE THE GRIN LENS

Let us consider the hybrid structure shown in Fig. 1, where we have assumed that the transverse dimension of the ZP placed on the input face of the GRIN lens is smaller than the lens' effective aperture,[5] so that the ZP acts as the entrance pupil of the system. A plane uniform wave of unit amplitude and wavelength λ is normally incident upon the ZP. There is an image of the point source (located at infinity) at the principal focal position of the ZP, which position, from Eq. (2), is given by

$$\int_0^{f_1} g(z')\,\mathrm{d}z' = \tan^{-1}\left(\frac{n_0 g_0 p}{2\lambda}\right). \tag{3}$$

Using the expression for the kernel of a GRIN medium with a transverse parabolic refractive index profile in polar coordinates[6]

$$K(r,r_0,\Omega,\Omega_0;z) = \frac{-\mathrm{i}kn_0}{2\pi H_1(z)}\exp\left(\mathrm{i}kn_0 z\right)\exp\left\{\frac{\mathrm{i}kn_0}{2H_1(z)}\left[\dot{H}_1(z)r^2 + H_2(z)r_0^2 - 2rr_0\cos(\Omega-\Omega_0)\right]\right\}, \tag{4}$$

the complex amplitude distribution produced by a single zone at a point $(r;z)$ inside the lens can be calculated by solving the Fresnel integral equation[6]

$$\psi(r;z) = \int_0^{2\pi}\int_{\rho_1}^{\rho_2} K(r,r_0,\Omega,\Omega_0;z)r_0\,\mathrm{d}r_0\,\mathrm{d}\Omega_0\,, \tag{5}$$

where r_0, Ω_0 and r, Ω are the radial and angular coordinates on the input face and on a plane z within the lens, respectively; $H_1(z), \dot{H}_1(z)$ and $H_2(z), \dot{H}_2(z)$ are the position and the slope at z of the axial and field rays, which can be written as[6]

$$H_1(z) = \left[g_0 g(z)\right]^{-1/2}\sin\left[\int_0^z g(z')\,\mathrm{d}z'\right] = -\left[g_0 g(z)\right]^{-1}\dot{H}_2(z)\,, \tag{6}$$

$$H_2(z) = \left[g_0/g(z)\right]^{1/2}\cos\left[\int_0^z g(z')\,\mathrm{d}z'\right] = \frac{g_0}{g(z)}\dot{H}_1(z)\,, \tag{7}$$

and the integration limits ρ_1 and ρ_2 correspond to the lower and upper ends of each zone, which assume the values

$$\rho_{\binom{1}{2}} = \left[\binom{N_1}{N_2}\frac{H_1(z)}{n_0 H_2(z)}\lambda\right]^{1/2} = \left[\binom{N_1}{N_2}\frac{a^2}{N}\right]^{1/2}, \tag{8}$$

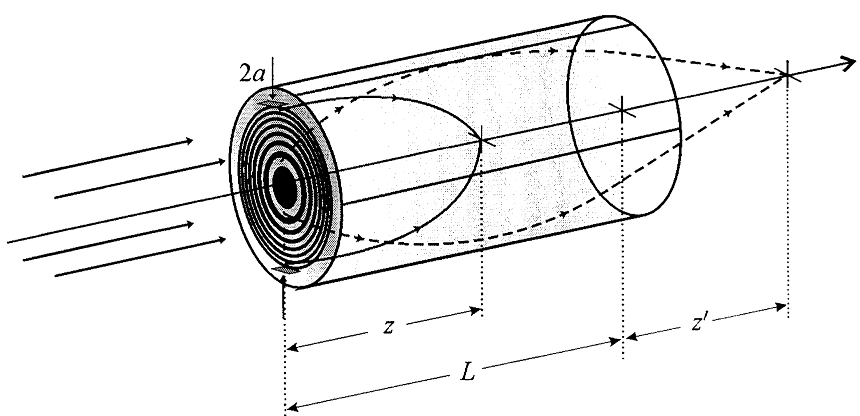

Fig. 1. Geometry for the calculation of the resolving power of the hybrid structure zone plate/GRIN lens.. The ZP, of outer radius a, is illuminated by a plane uniform wave front. The wave front focuses at planes z and $L + z'$ (with L being the total length of the GRIN lens), inside and outside the lens, respectively.

with a and N being the outer ZP radius and the total number of zones, respectively, and N_1 and N_2 being, for a positive Fresnel ZP of the amplitude, $N_1 = 2m$ and $N_2 = 2m+1$ (odd zones only); for a negative Fresnel ZP of the amplitude, $N_1 = 2m+1$ and $N_2 = 2m+2$ (even zones only); and, for a phase Fresnel ZP, $N_1 = m$ and $N_2 = m+1$ (all zones are retained). By integration in the angular coordinate in Eq. (5) we obtain

$$\psi(r;z) = \frac{-\mathrm{i}kn_0}{H_1(z)}\exp(\mathrm{i}kn_0 z)\exp\left[\frac{\mathrm{i}kn_0}{2H_1(z)}\dot{H}_1(z)r^2\right]\int_{\rho_a}^{\rho_b}\exp\left[\frac{\mathrm{i}kn_0}{2H_1(z)}H_2(z)r_0^2\right]\mathrm{J}_0\left[\frac{kn_0 r r_0}{H_1(z)}\right]r_0\mathrm{d}r_0 . \tag{9}$$

This equation has no analytical solution. However, if instead of considering the principal focus, we take into account the non-diffracted light at the ZP (order 0), which focuses at the Fourier transform plane[2] ($H_2 = 0$), the square modulus of Eq. (9) yields

$$I^{a+}(r;z) = \frac{4\pi^2 n_0^2}{\lambda^2 H_1^2(z)}\left(\sum_{m=0}^{\frac{1}{2}(N-1)}\frac{a^2}{\pi N^{1/2}C}\left\{(2m+1)^{1/2}\,\mathrm{J}_1\left[(2m+1)^{1/2}\frac{\pi C}{N^{1/2}}\right]-(2m)^{1/2}\,\mathrm{J}_1\left[(2m)^{1/2}\frac{\pi C}{N^{1/2}}\right]\right\}\right)^2 \tag{10}$$

for the positive Fresnel ZP of the amplitude,

$$I^{a-}(r;z) = \frac{4\pi^2 n_0^2}{\lambda^2 H_1^2(z)}\left(\sum_{m=0}^{\frac{1}{2}(N-2)}\frac{a^2}{\pi N^{1/2}C}\left\{(2m+2)^{1/2}\,\mathrm{J}_1\left[(2m+2)^{1/2}\frac{\pi C}{N^{1/2}}\right]-(2m+1)^{1/2}\,\mathrm{J}_1\left[(2m+1)^{1/2}\frac{\pi C}{N^{1/2}}\right]\right\}\right)^2 \tag{11}$$

for the negative Fresnel ZP of the amplitude, and

$$I^{f}(r;z) = \frac{4\pi^2 n_0^2}{\lambda^2 H_1^2(z)}\left[\sum_{m=0}^{N-1}\frac{a^2}{\pi N^{1/2}C}\left\{(m+1)^{1/2}\,\mathrm{J}_1\left[(m+1)^{1/2}\frac{\pi C}{N^{1/2}}\right]-m^{1/2}\,\mathrm{J}_1\left[m^{1/2}\frac{\pi C}{N^{1/2}}\right]\right\}\right]^2 \tag{12}$$

for the phase Fresnel ZP, where $C = 2n_0 ar/[\lambda H_1(z)]$.

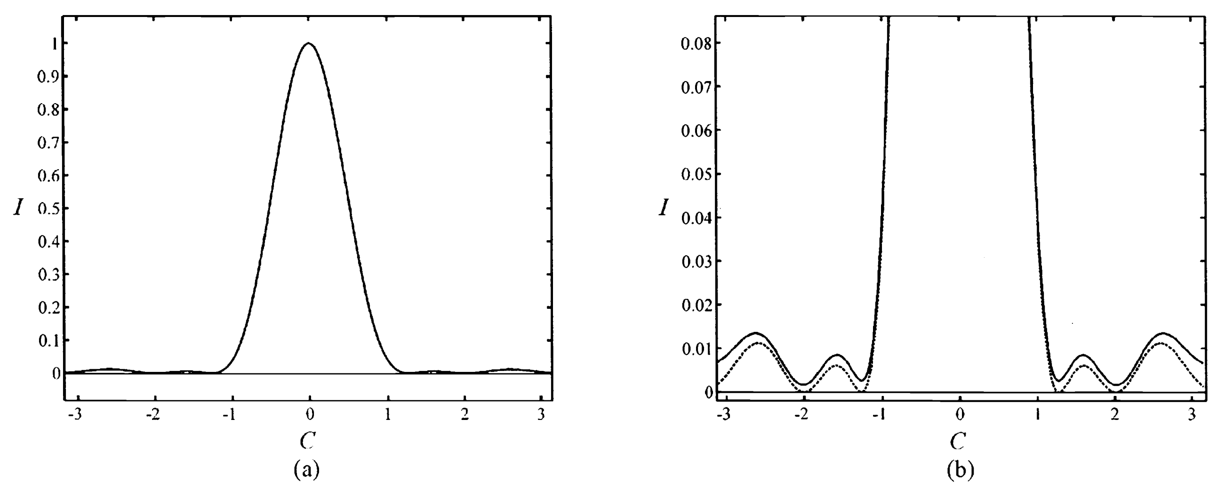

Fig. 2. Normalized irradiance distributions in terms of variable C for a positive Fresnel ZP of the amplitude with $N = 9$ within a Selfoc GRIN rod lens. (a) Irradiance at the Fourier transform plane. (b) Irradiance at the principal focal plane under the Fresnel (solid curve) and Fraunhofer (dashed curve) approximations. Calculations have been made for $\lambda = 1.3$ μm, outer radius $a = 0.18$ mm, ZP period $p = 7.2 \times 10^{-3}$ mm^2, $n_0 = 1.5$, $g_0 = 0.1$ mm^{-1}.

Figure 2(a) shows the normalized irradiance distribution at the Fourier transform plane as a function of variable C for a positive Fresnel ZP of the amplitude with number of zones $N = 9$ inside of a Selfoc lens (i.e. $g(z) = g_0$). Likewise, in Fig. 2(b) we present the normalized irradiance distribution at the principal focal plane f_1 as obtained either from the numerical resolution of Eq. (5) [solid line] or from Eq. (10) under the Fraunhofer approximation (dashed line). We can observe that the maximum value of the first side lobe of the solid curve is higher than the value that shows the dashed curve and, whereas the minima for the latter are always zero, the former curve assumes non-zero values. On the other hand, if we designate C_0 as the value of C for which the first minimum of irradiance takes place, we see that it assumes approximately the same value whether we use the Fresnel or the Fraunhofer approximation. C_0 varies depending on the type and number of zones of the ZP as depicted by Fig. 3. We see that the maximum discrepancy between the two approximations occur when the number of zones is very small. Thereby, for the positive and negative ZP $C_0 = 1.394$ for $N = 3$ and $C_0 = 0.892$ for $N = 2$ under the Fraunhofer approximation, whereas $C_0 = 1.356$ for $N = 3$ and $C_0 = 0.889$ for $N = 2$ under the Fresnel approximation. In both cases, as N increases the discrepancy becomes smaller and C_0 tends to the value for a conventional lens ($C_0 = 1.22$). For a phase ZP, C_0 coincides with that of the lens regardless of the number of zones.

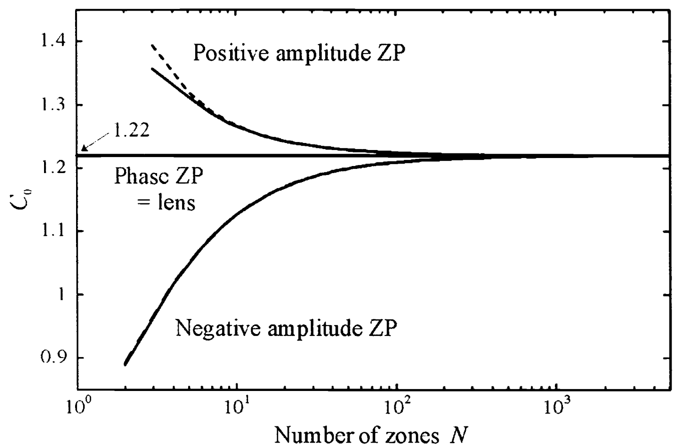

Fig. 3. C_0 as a function of the number of zones N for a positive and a negative ZP of the amplitude and a phase ZP. The solid curves are for the Fresnel approximation and the dashed curves for the Fraunhofer approximation.

By inspection of Fig. 2 we can conclude that the irradiance pattern produced by the hybrid structure describes Airy figures in which the radius of the central disk (Airy radius[7]) is given by

$$r_{\mathrm{Ai}} = \frac{C_0}{2} \frac{\lambda H_1(z)}{n_0 a}. \tag{13}$$

According to Rayleigh's criterion, and keeping in mind that a lateral displacement of a point source concerns a lateral shift of the irradiance pattern inside the GRIN lens proportional to the axial ray,[8] we can estimate the minimum transverse separation ζ between two points located on a plane a distance d from the ZP so that the hybrid system can resolve their images:

$$\zeta = C_0 \frac{\lambda d}{a}. \tag{14}$$

The resolving power of the hybrid system is then the same as that of a ZP in free space.

3. FOCAL POSITIONS OUTSIDE THE GRIN LENS

Let us now consider a practical case in which the image of a point source is formed at a distance z' from the output face of a GRIN lens of length L. The *ABCD* matrix that describes the propagation of light from the input face of the lens to a plane a distance z' from its output face is given by

$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} 1 & z' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} H_2(L) & \frac{H_1(L)}{n_0} \\ n_0 \dot{H}_2(L) & \dot{H}_1(L) \end{bmatrix} = \begin{bmatrix} H_2(L) + z' n_0 \dot{H}_2(L) & \frac{H_1(L)}{n_0} + z' \dot{H}_1(L) \\ n_0 \dot{H}_2(L) & \dot{H}_1(L) \end{bmatrix}, \tag{15}$$

where the GRIN medium ray-transfer matrix has been used.[6] Equation (15) can be written in the form of a kernel,[9] resulting

$$K^{\text{GRIN+fs}}(r,r_0,\Omega,\Omega_0;z') = \frac{-\mathrm{i}kn_0/(2\pi)}{\left[H_1(L)+n_0\dot{H}_1(L)z'\right]}\exp\left[\mathrm{i}k(n_0L+z')\right]$$

$$\times\exp\left\{\frac{\mathrm{i}\,kn_0/2}{H_1(L)+n_0\dot{H}_1(L)z'}\left[\left(H_2(L)+n_0\dot{H}_2(L)z'\right)r_0^2+\dot{H}_1(L)r^2-2rr_0\cos(\Omega-\Omega_0)\right]\right\}. \quad (16)$$

Substituting this expression in Eq. (5) and following the same process that we used to obtain Eq. (13), we find that the Airy radius for the system made up of the hybrid structure and free space is given by

$$r_{\text{Ai}} = \frac{C_0}{2}\frac{\lambda}{a}\left[\frac{H_1(L)}{n_0}+\dot{H}_1(L)\,z'\right], \quad (17)$$

which reduces to Eq. (13) when $z' = 0$.

In Fig. 4 we show a representation of Eq. (17) in terms of the system length L for the structure described in Fig. 2 when the object point is placed at several distances d from the ZP. The distance z' from the output face of the lens to the principal focal position outside the GRIN lens is given by[3]

$$z' = \frac{(2m\lambda d-p)H_1(L)-n_0dpH_2(L)}{n_0\left[(p-2m\lambda d)\dot{H}_1(L)+n_0dp\dot{H}_2(L)\right]}. \quad (18)$$

We vary L from zero up to a value that cancels z'. When $L = 0$ we have the Airy radius of the ZP. When $z' = 0$ we obtain a minimum value for the Airy radius. We can observe that the Airy radius monotonically decreases with increasing L, being the decrease bigger for short illumination distances d. The curves for the different illumination distances are parallel and get closer to one another as the wave-front curvature diminishes.

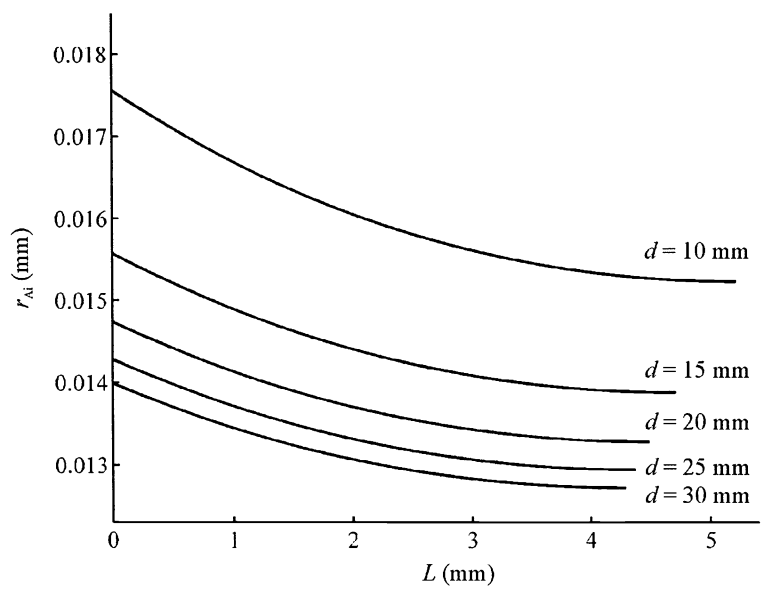

Fig. 4. Airy radius on the principal focal plane outside the GRIN lens as a function of the lens length for several illumination distances d. L varies from zero (when the hybrid structure reduces to a ZP) to a length that makes the outside distance $z' = 0$. Calculations have been made for the data specified in Fig. 2 and $C_0 = 1.27$ ($N = 9$).

CONCLUSIONS

Irradiance patterns inside and outside a hybrid system composed of a GRIN lens and a Fresnel zone plate of the amplitude and phase have been studied in the Fresnel and Fraunhofer approximations. In both cases, we have seen that the irradiance distribution describes Airy figures in which the radius of the central bright spot tends to that produced by a conventional lens as the number of zones of the ZP increases. The irradiance of the side lobes is higher for the Fresnel approximation and the minima of irradiance are non-zero. On the other hand, for a small number of zones, negative zones plates of the amplitude resolve the image of an object better that other zone-plate configurations. Finally, we have proved that with hybrid structures we can form an image of an object point with a smaller Airy radius than if we only used a ZP.

ACKNOWLEDGMENT

This study was supported by the Ministerio de Ciencia y Tecnología, Spain, under contract TIC-2003-03041.

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