Fractional Talbot effect in a tapered gradient-index medium: unit cell

María Teresa Flores-Arias

Laboratorio de Optica, Departamento de Física Aplicada, Facultad de Física y Escuela de Optica y Optometría, Universidade de Santiago de Compostela, Campus Sur, E15706, Santiago de Compostela, Galicia, Spain

Carlos R. Fernandez-Pousa

Departamento de Ciencia y Tecnología de Materiales, Division de Optica, Edificio la Galia, Universitat Miguel Hernandez, E03202 Elx, Alacant, Spain

María Victoria Perez, Carmen Bao, and Carlos Gómez-Reino

Laboratorio de Optica, Departamento de Física Aplicada, Facultad de Física y Escuela de Optica y Optometría, Universidade de Santiago de Compostela, Campus Sur, E15706, Santiago de Compostela, Galicia, Spain

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A generalization of the fractional Talbot effect to the case of a tapered gradient-index medium for uniform illumination is considered. A unit cell of the fractional Talbot image contains the superposition of unit cell images of the periodic object. © 2000 Optical Society of America [S0740-3232(00)00406-3]

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1. INTRODUCTION

The integer and fractional Talbot effect in a homogeneous medium is well known in optics and has received wide attention.[1,2] The integer Talbot effect has also been studied in transverse quadratic-index media,[3,4] and recently authors have described the integer Talbot effect in a tapered gradient-index (GRIN) medium for nonuniform and uniform illumination.[5]

The purpose of this paper is to generalize the fractional Talbot effect to a tapered GRIN medium when a periodic object located at the input of the medium is illuminated by a coherent uniform light beam as a logical continuation of a previous paper.[5] The study will be restricted to the one-dimensional transverse case, but extension of the analysis to the two-dimensional case is straightforward.

2. FRACTIONAL TALBOT EFFECT

Let us consider a tapered GRIN medium characterized by a transverse parabolic refractive index modulated by an axial index and whose refractive index is given by

$$n^2(x, z) = n_0{}^2[1 - g^2(z)x^2], \tag{1}$$

where n_0 is the index at the z optical axis and $g(z)$ the taper function that describes the evolution of the transverse index along the z axis.

We assume a one-dimensional periodic object of infinite dimension located at the input of the GRIN medium and whose transmission function will be represented as

$$T(x_0) = \sum_{m=-\infty}^{+\infty} a_m \exp\left(-i\frac{2\pi m x_0}{p}\right), \tag{2}$$

where p is the spatial period and a_m is the amplitude of the mth harmonic.

When the hybrid optical structure formed by the periodic object and the tapered GRIN medium is illuminated by a coherent uniform beam [Fig. 1(a)], the complex amplitude distribution on the periodic object located at $z = 0$ can be written as

$$\phi(x_0) = T(x_0)\psi_0(x_0), \tag{3}$$

where

$$\psi_0(x_0) = \frac{1}{\sqrt{d}}\exp\left(i\frac{\pi x_0{}^2}{\lambda d}\right) \tag{4}$$

is the complex amplitude distribution due to a cylindrical wave front of wavelength λ and d is the radius of curvature of the wave front.

The complex amplitude distribution $\phi(x; z)$ in the tapered GRIN medium at $z > 0$ is given by the integral equation[5]

$$\phi(x; z) = \int_{-\infty}^{+\infty} \phi(x_0) K(x, x_0; z)\mathrm{d}x_0, \tag{5}$$

where K is the one-dimensional optical propagator of this medium expressed as

$$K(x, x_0; z) = \left[\frac{n_0}{i\lambda H_1(z)}\right]^{1/2} \times \exp(ikn_0 z)\exp\left\{i\frac{kn_0}{2H_1(z)}[x^2\dot{H}_1(z) + x_0{}^2 H_2(z) - 2xx_0]\right\}, \tag{6}$$

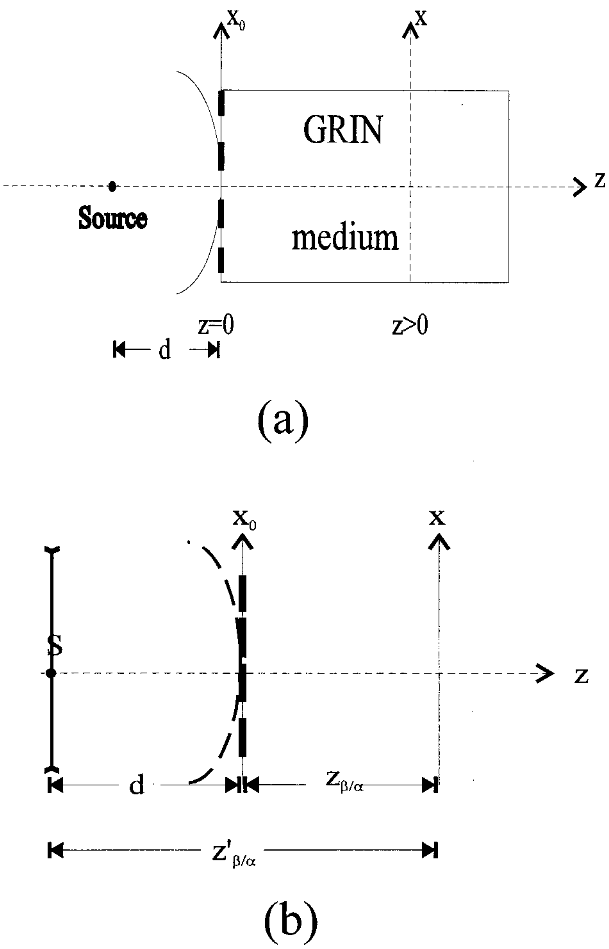

Fig. 1. Geometry for the evaluation of (a) the complex amplitude distribution in a tapered GRIN medium due to a periodic object located at $z = 0$ and illuminated by a cylindrical uniform beam and (b) the equivalent optical system for the fractional Talbot effect under divergent uniform illumination.

where H_1, H_2, and $\dot{H}_1$, $\dot{H}_2$ are the position and the slope, respectively, of the axial and field rays at z and the dot is the derivative with respect to z.

Substituting Eqs. (3) and (4) into Eq. (5) and integrating, we have

$$\phi(x; z) = \frac{1}{\sqrt{dF(z)}} \exp(ikn_0 z) \exp\left[i \frac{kn_0 \dot{F}(z)}{2F(z)} x^2\right] \times \sum_{m=-\infty}^{+\infty} a_m \exp\left[-i \frac{2\pi m x}{pF(z)}\right] \times \exp\left[-i\pi m^2 \frac{\lambda H_1(z)}{n_0 p^2 F(z)}\right], \tag{7}$$

where

$$\dot{F}(z) = \dot{H}_2(z) + \frac{\dot{H}_1(z)}{n_0 d}. \tag{8}$$

The second-term exponential of the summation in Eq. (7) is of basic importance to self-imaging. Periodic repetition along the optical z axis of lateral complex amplitude distribution occurs for the Talbot condition given by

$$\frac{\lambda H_1(z_{\beta/\alpha})}{n_0 p^2 F(z_{\beta/\alpha})} = \frac{\beta}{\alpha}, \tag{9}$$

where α and β are integers and β/α is referred to as the self-image number. In particular, the fractional Talbot effect is obtained for α and β co-prime integers, that is, for β/α fractional numbers.

Equation (9), which determines the fractional Talbot condition for a tapered GRIN medium, may also be expressed as

$$\frac{\beta p^2}{\alpha \lambda d_0{}^2} = \frac{1}{d} - \frac{1}{z'_{\beta/\alpha}}, \tag{10}$$

where

$$z'_{\beta/\alpha} = F(z_{\beta/\alpha}) d = d + z_{\beta/\alpha}, \tag{11a}$$

$$d_0{}^2 = \frac{d^2 H_1(z_{\beta/\alpha})}{n_0 z_{\beta/\alpha}}, \tag{11b}$$

$$z_{\beta/\alpha} = [F(z_{\beta/\alpha}) - 1] d. \tag{11c}$$

Therefore the fractional Talbot effect can be considered the equivalent effect of an apparent lens of multifocal length $f_{\beta/\alpha} = (\lambda d_0{}^2 \alpha)/(p^2 \beta)$ situated at the linear source that illuminates a periodic object, as shown in Fig. 1(b), and whose lateral magnification is written as

$$M_t(z_{\beta/\alpha}) = \frac{z'_{\beta/\alpha}}{d} = F(z_{\beta/\alpha}). \tag{12}$$

The fractional self-image distances correspond to the length $z_{\beta/\alpha}$ of a tapered GRIN medium for which the input complex amplitude distribution is periodically repeated along the z axis. The axial localization of fractional self-images can be obtained from the fractional Talbot condition, if we take into account that H_1 and H_2 are given by

$$H_1(z) = [g_0 g(z)]^{-1/2} \sin\left[\int_0^z g(z') \mathrm{d}z'\right], \tag{13}$$

$$H_2(z) = \left[\frac{g_0}{g(z)}\right]^{1/2} \cos\left[\int_0^z g(z') \mathrm{d}z'\right], \tag{14}$$

where g_0 is the value of $g(z)$ at $z = 0$.

Substituting Eqs. (13) and (14) into Eq. (9), we find that axial localization of fractional self-images is given by

$$\int_0^{z_{\beta/\alpha}} g(z') \mathrm{d}z' = \tan^{-1}\left(\frac{n_0 g_0 d p^2 \beta}{\lambda d \alpha - p^2 \beta}\right). \tag{15}$$

Figure 2 shows the axial position of self-images versus self-image number for a tapered GRIN medium with a divergent linear taper function given by

$$g(z) = \frac{g_0}{1 - z/L}, \tag{16}$$

where L is the distance from $z = 0$ to the common apex of the equi-index lines of the refractive-index profile.

The axial location of self-images is given by

$$z_{\beta/\alpha} = L\left\{\exp\left[\frac{1}{g_0} \tan^{-1}\left(\frac{n_0 g_0 d p^2 \beta}{d \alpha \lambda - p^2 \beta}\right)\right] - 1\right\}. \tag{17}$$

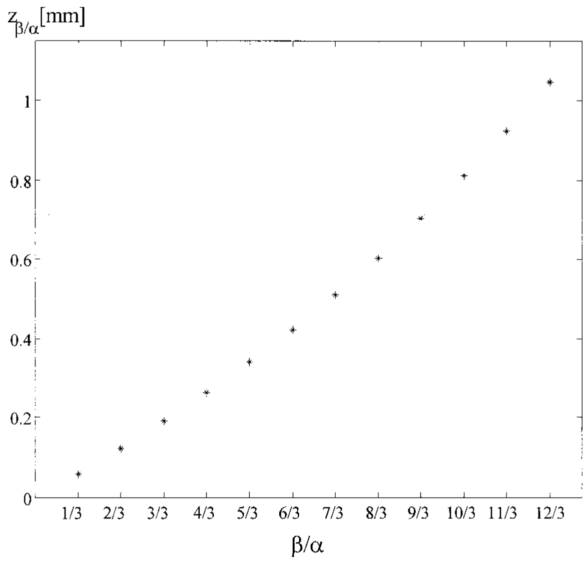

Fig. 2. Axial position of self-images versus self-image number for a tapered GRIN medium with a divergent linear taper function. Calculations have been made for $n_0 = 1.5$, $g_0 = 0.01\,\mathrm{mm}^{-1}$, $p = 9\,\mu\mathrm{m}$, $\lambda = 0.7\,\mu\mathrm{m}$, $L = 1\,\mathrm{mm}$, $d = 15\,\mathrm{mm}$, and $\alpha = 3$.

Figure 2 indicates that the interval between consecutive fractional self-images increases with β/α. For $\beta/\alpha = 3/3, 6/3, 9/3$, and $12/3$, the first four integer self-images are obtained.

Finally, if the periodic object is illuminated by a Gaussian beam (nonuniform illumination), the fractional Talbot condition, given by Eq. (9), becomes

$$\frac{\lambda\, \mathrm{Re}[F^g(z_{\beta/\alpha})]H_1(z_{\beta/\alpha})}{n_0 p^2 |F^g(z_{\beta/\alpha})|^2} = \frac{\beta}{\alpha}, \tag{18}$$

where

$$F^g(z_{\beta/\alpha}) = \frac{U(0)H_1(z_{\beta/\alpha})}{n_0} + H_2(z_{\beta/\alpha}), \tag{19}$$

$U(0)$ is the complex curvature of the Gaussian beam at $z = 0$, and $\mathrm{Re}[F^g(z_{\beta/\alpha})]$ is the real part of $F^g(z_{\beta/\alpha})$.

3. UNIT CELL

To explain the above results, we consider an input object of period p, consisting of a Dirac comb with the transmission function given by

$$T_c(x_0) = \sum_{m=-\infty}^{+\infty} \delta(x_0 - mp) = \frac{1}{p}\sum_{m=-\infty}^{+\infty} \exp\left(-i\frac{2\pi m x_0}{p}\right), \tag{20}$$

where δ is the Dirac delta function.

Inserting Eq. (20) into Eq. (3), we find that the complex amplitude distribution at fractional Talbot distances becomes

$$\phi_c(x;\, z_{\beta/\alpha}) = \frac{1}{p\sqrt{dF(z_{\beta/\alpha})}}\exp(ikn_0 z_{\beta/\alpha})\exp\left[i\frac{kn_0\dot{F}(z_{\beta/\alpha})}{2F(z_{\beta/\alpha})}x^2\right] \times \sum_{m=-\infty}^{+\infty}\exp\left[-i\frac{2\pi m x}{pF(z_{\beta/\alpha})}\right]\exp\left(-i\frac{\pi m^2\beta}{\alpha}\right), \tag{21}$$

where Eqs. (5) and (8) have been used.

After a long but straightforward calculation, Eq. (21) can be rewritten as[2]

$$\phi_c(x;\, z_{\beta/\alpha}) = \left[\frac{F(z_{\beta/\alpha})}{\alpha d}\right]^{1/2}\exp(ikn_0 z_{\beta/\alpha})\exp\left[i\frac{kn_0\dot{F}(z_{\beta/\alpha})}{2F(z_{\beta/\alpha})}x^2\right] \times \sum_{m=-\infty}^{+\infty}\delta\left[x' - \frac{mpF(z_{\beta/\alpha})}{\alpha}\right]A(m;\, \alpha,\, \beta), \tag{22}$$

where

$$x' = x + \tfrac{1}{2}pF(z_{\beta/\alpha})e_\beta, \tag{23}$$

with $e_\beta = 0(1)$ if β is even (odd) and A denoting pure phase factors given by (see Appendix A)

$$A(m;\, \alpha,\, \beta) = \frac{1}{\sqrt{\alpha}}\sum_{s=1}^{\alpha}\exp\left\{i\frac{\pi}{\alpha}\left[2s\left(m + \frac{\alpha e_\beta}{2}\right) - \beta s^2\right]\right\}. \tag{24}$$

From Eq. (22) it follows that the summation at $z_{\beta/\alpha}$ may be regarded as a replica of the input Dirac comb, with spacing $pF(z_{\beta/\alpha})/\alpha$, weighted by the phase factors A, since

$$\sum_{m=-\infty}^{+\infty}\delta\left[x' + pF(z_{\beta/\alpha}) - \frac{mpF(z_{\beta/\alpha})}{\alpha}\right]A(m;\, \alpha,\, \beta) = \sum_{n=-\infty}^{+\infty}\delta\left[x' - \frac{npF(z_{\beta/\alpha})}{\alpha}\right]A(n;\, \alpha,\, \beta). \tag{25}$$

Then the fractional Talbot image reproduces periodically a unit cell of period $pF(z_{\beta/\alpha})$ containing α weighted images of the unit cell of the original Dirac comb and irradiance proportional to $|F(z_{\beta/\alpha})|/\alpha d$. If β is even, the Talbot image is centered at $x = 0$, and if β is odd, the Talbot image is laterally shifted and centered at $x = -[pF(z_{\beta/\alpha})]/2$.

After that, we can return to the general periodic object. The transmission function, given by Eq. (2), is also expressed as

$$T(x_0) = \int_{-\infty}^{+\infty} t(\eta)\sum_{m=-\infty}^{+\infty}\delta(\eta - x_0 + mp)\mathrm{d}\eta = \sum_{m=-\infty}^{+\infty} t(x_0 - mp), \tag{26}$$

where t is the transmission function in the unit cell of the periodic object.

Substituting Eq. (26) into Eq. (5), we find that the complex amplitude distribution in the tapered GRIN medium is given by

$$\phi(x;\, z) = \int_{-\infty}^{+\infty} t(\eta)\exp\left\{ikn_0\dot{F}(z)\eta\left[x - \frac{\eta F(z)}{2}\right]\right\} \times \phi_c[x - \eta F(z);\, z]\mathrm{d}\eta, \tag{27}$$

where

$$\phi_c(x;\, z) = \int_{-\infty}^{+\infty} K(x,\, x_0;\, z)\psi_0(x_0)\sum_{m=-\infty}^{+\infty}\delta(x_0 - mp)\mathrm{d}x_0. \tag{28}$$

Equation (27) at fractional Talbot distances reduces to

$$\phi(x;\, z_{\beta/\alpha}) = \left[\frac{1}{\alpha dF(z_{\beta/\alpha})}\right]^{1/2}\exp(ikn_0 z_{\beta/\alpha}) \times \exp\left[i\frac{kn_0\dot{F}(z_{\beta/\alpha})}{2F(z_{\beta/\alpha})}x^2\right] \times \sum_{m=-\infty}^{+\infty} t\left\{\frac{1}{F(z_{\beta/\alpha})}\left[x' - \frac{mpF(z_{\beta/\alpha})}{\alpha}\right]\right\} \times A(m;\, \alpha,\, \beta), \tag{29}$$

where Eq. (22) has been used.

Equation (29) is the central result of the present analysis. From a comparison of this equation and Eq. (22), it follows that at $z_{\beta/\alpha}$ each unit cell of a Talbot image of irradiance $1/\alpha d|F(z_{\beta/\alpha})|$ consists of α weighted images of the unit cell of the original periodic object with scaling factor $pF(z_{\beta/\alpha})/\alpha$. When β is even, the fractional Talbot image of period $pF(z_{\beta/\alpha})$ is centered with respect to the object, and when β is odd, the image presents a transverse shift. Likewise, the period of fractional self-images carries information about the transverse magnification of the periodic object due to uniform illumination, given by $F(z_{\beta/\alpha})$, which coincides with Eq. (12).

For $\alpha = 1$, where β/α is an integer and $A(m;1,\beta) = 1$, Eq. (29) becomes

$$\phi(x,\, z_\beta) = \left[\frac{1}{dF(z_\beta)}\right]^{1/2}\exp(ikn_0 z_\beta)\exp\left[i\frac{kn_0\dot{F}(z_\beta)}{2F(z_\beta)}x^2\right] \times \sum_{m=-\infty}^{+\infty} a_m\exp\left[-i\frac{2\pi mx'}{pF(z_\beta)}\right], \tag{30}$$

and integers Talbot images are obtained.[5]

4. CONCLUSIONS

A generalization of the fractional Talbot effect to a tapered GRIN medium has been considered. At distances $z_{\beta/\alpha}$, each unit cell of a fractional Talbot image consists of α weighted images of the unit cell of the original periodic object. For β even, fractional Talbot images are centered with respect to the original object, and for β odd, images present transverse shifts.

APPENDIX A

We wish to prove that A is a pure phase factor. We must show that

$$|A(m;\, \alpha,\, \beta)| = 1. \tag{A1}$$

From Eq. (24) we have

$$|A(m;\, \alpha,\, \beta)|^2 = \frac{1}{\alpha}\sum_{s=1}^{\alpha}\sum_{t=1}^{\alpha}\exp\left\{i\frac{\pi}{\alpha}\left[2(s - t)\left(m + \frac{\alpha e_\beta}{2}\right) - \beta(s - t)(s + t)\right]\right\}. \tag{A2}$$

Under the change of variable

$$u = s - t, \tag{A3}$$

Eq. (A2) becomes

$$|A(m;\, \alpha,\, \beta)|^2 = \frac{1}{\alpha}\sum_{s=1}^{\alpha}\sum_{u=s-\alpha}^{s-1}\exp\left\{i\frac{\pi}{\alpha}\left[2u\left(m + \frac{\alpha e_\beta}{2}\right) - \beta(2s - u)u\right]\right\}. \tag{A4}$$

This equation can be rewritten as

$$|A(m;\, \alpha,\, \beta)|^2 = \frac{1}{\alpha}\sum_{u=0}^{\alpha-1}\exp\left\{i\frac{\pi}{\alpha}\left[2u\left(m + \frac{\alpha e_\beta}{2}\right) + \beta u^2\right]\right\} \times \sum_{s=1}^{\alpha}\exp\left\{-i\frac{2\pi\beta us}{\alpha}\right\}, \tag{A5}$$

where the following relationship has been used:

$$\exp\left\{i\frac{\pi}{\alpha}\left[2u\left(m + \frac{\alpha e_\beta}{2}\right) - \beta(2s - u)u\right]\right\} = \exp\left\{i\frac{\pi}{\alpha}\left[2(u + \alpha)\left(m + \frac{\alpha e_\beta}{2}\right) - \beta(2s - u - \alpha)(u + \alpha)\right]\right\}. \tag{A6}$$

A second summation into Eq. (A5) is given by

$$\sum_{s=1}^{\alpha}\exp\left\{-i\frac{2\pi\beta us}{\alpha}\right\} = \alpha\delta_{uo}, \tag{A7}$$

where δ_{uo} is the Kronecker delta.
Then Eq. (A5) reduces to

$$|A(m;\, \alpha,\, \beta)|^2 = \frac{1}{\alpha}\sum_{u=0}^{\alpha-1}\exp\left\{i\frac{\pi}{\alpha}\left[2u\left(m + \frac{\alpha e_\beta}{2}\right) + \beta u^2\right]\right\}\alpha\delta_{uo} = 1. \tag{A8}$$

Thus A is a pure phase factor.

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The author can be reached at the address on the title page or by telephone at 34-981-563100 ext. 13512, by fax at 34-981-521984, or by e-mail at facgrc@usc.es.

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