

Integral Equation Methods for Scattering by One-Dimensional Rough Surfaces*

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Abstract. We consider the Dirichlet boundary value problem for the Helmholtz equation in a non-locally perturbed half-plane which arises in a study of time harmonic acoustic scattering of an incident field by a sound-soft, infinite rough surface where the total field vanishes. We propose a second kind boundary integral equation formulation of this problem, the novel feature of which is that it uses a half-plane Green's function in place of the standard free-field fundamental solution. This has the advantage that the integral equation is well posed in the space of bounded continuous functions and in the L^p spaces, $p \in [1, \infty]$, and that results can be obtained on convergence of the finite section method (truncating the range of integration) and on convergence of a banded matrix iteration algorithm. Numerical results suggest the theoretically predicted advantages are seen in practice.

1. Introduction. We consider the two-dimensional Dirichlet boundary value problem for the Helmholtz equation $\Delta u + k^2 u = 0$ in a non-locally perturbed half-plane. This problem arises in a study of time-harmonic acoustic scattering of an incident field by a sound-soft, infinite rough surface where the total field vanishes; the same problem models two-dimensional electromagnetic scattering by a perfectly conducting, infinite, rough surface in one of the polarization cases (see e.g. Petit [22], DeSanto and Brown [8]).

The problem of scattering of waves from rough surfaces has been of interest to physicists, engineers, and applied mathematicians for many years because of its large number of applications in optics, acoustics, radiowave propagation, and radar techniques. It is important, for example, in the study of thin coatings in optics and, at a very different wavelength scale, in the propagation of radar or radio waves over the surface of the sea or irregular terrain.

A variety of approximate and numerical techniques are employed to compute solutions to rough surface scattering problems (Ogilvy [21], Voronovich [27]). Particularly popular to calculate accurate solutions, capturing all the multiple reflections which can take place, are boundary integral equation methods (see e.g. [9, 12, 14, 15, 20, 26, 28]).

It is usual to assume that the standard boundary integral equation formulations for problems of scattering by bounded obstacles remain valid when the boundary is unbounded. For example, a first kind integral equation for scattering by bounded obstacles on which the total field vanishes, derived from Green's representation theorem, is used in the simulations in Tsang *et al.* [26]. This formulation is that

$$u^t(x) = u^i(x) + \int_{\Gamma} \Phi(x, y) \frac{\partial u^t}{\partial n}(y) ds(y), \quad x \in \overline{D}, \quad (1)$$

where Γ is the unbounded rough surface, D is the (unbounded) region of propagation above Γ , and u^i and u^t are the incident field and total field, both solutions of the Helmholtz equation in D . In this equation, and subsequently, $n(y)$ is the unit normal at y , directed into D , and

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$\Phi(x, y)$ is the standard free space Green's function, defined by $\Phi(x, y) := \frac{i}{4}H_0^{(1)}(k|x - y|)$, where $H_0^{(1)}$ is the Hankel function of the first kind of order zero. Since u^t vanishes on Γ , (1) is a first kind integral equation for the unknown $\partial u^t / \partial n$ on Γ . In practice (e.g. in Tsang *et al.* [26]), Γ is truncated to a finite region (or arc in 2D) and then a boundary element method is applied.

From the mathematician's point of view, the above procedure is open to criticism. It is not clear that Green's representation theorem still applies to derive the integral equation if the boundary is unbounded (especially for an incident plane wave – but see DeSanto and Martin [10] for partial results in this direction). Further, it is not clear if the integral equation is well-posed, in particular whether it is uniquely solvable: this is even less clear once a truncation has taken place.

There are also interesting questions to ask about the performance of numerical algorithms. If, in the 2D case we consider in this paper (so that the surface is one-dimensional), we apply a discretisation with step-size h , and our truncated arc has length $2A$, do our numerical algorithms remain stable in the limit that $h \rightarrow 0$ and $A \rightarrow \infty$? (Both limits will be necessary to obtain accurate results.) As A increases, do we have to decrease h to maintain the same discretisation error, or can we achieve a desired level of discretisation error with h fixed (e.g. as 1/10th wavelength), independently of A ?

Finally, the limits $h \rightarrow 0$ and $A \rightarrow \infty$ lead to large dense systems of equations to solve. In the above cited papers much attention is applied to developing effective matrix compression and iterative schemes. It is well known that, for wave scattering problems, where the kernel of the integral operator is oscillatory, it is a challenging problem to develop and analyse such schemes.

In this paper we survey recent work at Brunel addressing some of the above questions. We propose, specifically for the two-dimensional Dirichlet problem, a precise mathematical formulation as a boundary value problem, an integral equation formulation which we show to be equivalent, and discuss briefly effective numerical schemes, deriving inspiration in particular from the paper by Tsang *et al* cited above. A key idea, which is crucial to all the analysis, is that we propose to use, in place of the free space Green's function, the Dirichlet Green's function for the upper half-plane as our fundamental solution. This decays much faster as $|x - y| \rightarrow \infty$ with $x, y \in \Gamma$, and has the effect that the integral operators become bounded (though not compact): in contrast the corresponding integral operators using the standard free space Green's function are not bounded on any standard function space. This more rapid decay also makes the integral operators more diagonal and this has computational advantages as we see in the numerical results.

2. The boundary value problem and integral equation formulation. Let $C^{1,1}(\mathbb{R})$ denote the Hölder space $C^{1,1}(\mathbb{R}) := \{\phi \in C^1(\mathbb{R}) \mid \|\phi\|_{C^{1,1}(\mathbb{R})} := \|\phi\|_\infty + \|\phi'\|_\infty + \sup_{s,t \in \mathbb{R}, s \neq t} |\phi'(s) - \phi'(t)|/|s - t| < \infty\}$. Given $f \in C^{1,1}(\mathbb{R})$ with $f_- := \inf_{x_1 \in \mathbb{R}} f(x_1) > 0$, define $D := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 > f(x_1)\}$ and let $\Gamma := \{(x_1, f(x_1)) \mid x_1 \in \mathbb{R}\}$ denote the boundary of D . We consider the problem of scattering of a time harmonic wave u^i , a solution of the Helmholtz equation $\Delta u^i + k^2 u^i = 0$ in D , incident on the infinite boundary Γ . We assume that $k > 0$ and restrict our attention to the case when the total field vanishes on the boundary, so that the scattered field u , also a solution of the Helmholtz equation in D , satisfies the Dirichlet boundary condition $u = -u^i$ on Γ . In order for the problem to have a unique solution, the scattered field u is required to satisfy the so-called *upward propagating radiation condition* proposed in [2] and [4]: that, for some $h > f_+ := \sup_{x_1 \in \mathbb{R}} f(x_1)$ and $\phi \in L_\infty(\Gamma_h)$,

$$u(x) = 2 \int_{\Gamma_h} \frac{\partial \Phi(x, y)}{\partial y_2} \phi(y) ds(y), \quad x \in U_h. \quad (2)$$

In this equation and hereafter, for $h \in \mathbb{R}$, $\Gamma_h := \{x \in \mathbb{R}^2 \mid x_2 = h\}$ and $U_h := \{x \in \mathbb{R}^2 \mid x_2 > h\}$.

Let $BC(\Gamma)$ denote the set of functions bounded and continuous on Γ , a Banach space under the norm $\|\cdot\|_{\infty,\Gamma}$ defined by $\|\phi\|_{\infty,\Gamma} := \sup_{x \in \Gamma} |\phi(x)|$. Assuming that $g := -u^i|_{\Gamma} \in BC(\Gamma)$, the case for the usual incident fields of interest including the incident plane wave, the above scattering problem can be formulated as the following boundary value problem for the scattered field u .

Problem (P): Given $g \in BC(\Gamma)$, determine $u \in C^2(D) \cap C(\bar{D})$ such that: (i) u is a solution of the Helmholtz equation $\Delta u + k^2 u = 0$ in D ; (ii) $u = g$ on Γ ; (iii) for some $\beta \in \mathbb{R}$, $\sup_{x \in D} x_2^\beta |u(x)| < \infty$; (iv) u satisfies the upward propagating radiation condition (2).

It is shown in [5] that this boundary value problem has exactly one solution.

Let $G(x, y) = \Phi(x, y) - \Phi(x, y')$ for $x, y \in \bar{U}_0$, $x \neq y$, where $y = (y_1, y_2)$, $y' = (y_1, -y_2)$. Then $G(x, y)$ is the Green's function for the operator $\Delta + k^2$ in the upper half-plane U_0 which satisfies the Dirichlet boundary condition

$$G(x, y) = 0, \quad x \in \Gamma_0, \quad y \in \bar{U}_0, \quad x \neq y. \quad (3)$$

It was proposed in [16] and in [29], to seek a solution to Problem (P) in the form of the combined double- and single-layer potential

$$u(x) = \int_{\Gamma} \left(\frac{\partial G(x, y)}{\partial n(y)} - i\eta G(x, y) \right) \psi(y) ds(y), \quad x \in D, \quad (4)$$

for some $\psi \in BC(\Gamma)$, where the coupling parameter η is chosen such that $\Re(\eta k) > 0$. (This echoes the standard combined layer potential approach (with $\Phi(x, y)$ as the fundamental solution) in the bounded obstacle case – see [7].) The combined layer potential in (4) exists for all $\psi \in BC(\Gamma)$ in view of the rapid decay of $G(x, y)$ as $|x_1 - y_1| \rightarrow \infty$ with $x_2, y_2 = O(1)$, expressed in the bound

$$|G(x, y)|, |\nabla_y G(x, y)| \leq C(1 + x_2)(1 + y_2)|x - y|^{-3/2}, \quad x, y \in \bar{U}_0, \quad x \neq y, \quad (5)$$

where the constant $C > 0$ depends only on k .

Theorem 1 [29] *The double-layer potential (4) satisfies Problem (P), with $\beta = -1/2$, provided $\psi \in BC(\Gamma)$ satisfies the boundary integral equation*

$$\psi(x) = 2 \int_{\Gamma} \left(-\frac{\partial G(x, y)}{\partial n(y)} + i\eta G(x, y) \right) \psi(y) ds(y) + 2g(x), \quad x \in \Gamma. \quad (6)$$

Defining $\tilde{\psi}, \tilde{g} \in BC(\mathbb{R})$ by

$$\tilde{\psi}(s) := \psi(s, f(s)), \quad \tilde{g}(s) := g(s, f(s)), \quad s \in \mathbb{R}, \quad (7)$$

and parametrizing the integral in (6) in the obvious way we obtain the following integral equation problem: find $\tilde{\psi} \in BC(\mathbb{R})$ such that

$$\tilde{\psi}(s) + 2 \int_{-\infty}^{+\infty} \left(\frac{\partial G(x, y)}{\partial n(y)} - i\eta G(x, y) \right) \tilde{\psi}(t) \sqrt{1 + [f'(t)]^2} dt = 2\tilde{g}(s), \quad s \in \mathbb{R}, \quad (8)$$

where $x = (s, f(s))$, $y = (t, f(t))$. Define the kernel k_f by

$$k_f(s, t) := -2 \left(\frac{\partial G(x, y)}{\partial n(y)} - i\eta G(x, y) \right) \sqrt{1 + [f'(t)]^2}, \quad s, t \in \mathbb{R}, \quad s \neq t, \quad (9)$$

with $x = (s, f(s))$, $y = (t, f(t))$, and define the integral operator K_f by

$$K_f \phi(s) := \int_{-\infty}^{+\infty} k_f(s, t) \phi(t) dt, \quad s \in \mathbb{R}, \quad (10)$$

where the subscript f indicates the dependence on the boundary. Then the integral equation (8) can be written in operator form as

$$(I - K_f)\tilde{\psi} = 2\tilde{g}. \quad (11)$$

That (11) has at most one solution follows from the uniqueness result for the original boundary value problem and from a uniqueness result for an interior problem in the waveguide region $U_0 \setminus D$, with homogeneous impedance and Dirichlet boundary conditions on ∂D and ∂U_0 , respectively — see [29] and cf. [5].

For some $c_1, c_2 > 0$ let B be defined by

$$B := \{f \in C^{1,1}(\mathbb{R}) \mid f(s) \geq c_1, s \in \mathbb{R} \text{ and } \|f\|_{C^{1,1}(\mathbb{R})} \leq c_2\}.$$

Then, since (11) has at most one solution for every $f \in B$, the following existence and uniqueness result [29] for the (equivalent) integral equations (6) and (8) follows from Theorem A.1 in the Appendix.

Theorem 2 *Suppose that $\Re(\eta k) > 0$. Then, for all $f \in B$, the operator $I - K_f : BC(\mathbb{R}) \rightarrow BC(\mathbb{R})$ is bijective (and so boundedly invertible) with*

$$\sup_{f \in B} \|(I - K_f)^{-1}\|_{BC(\mathbb{R}) \rightarrow BC(\mathbb{R})} < \infty. \quad (12)$$

Thus the integral equation (8) has exactly one solution for every $f \in B$ and $g \in BC(\Gamma)$, with

$$\|\tilde{\psi}\|_\infty \leq C \|\tilde{g}\|_\infty$$

for some constant $C > 0$ depending only on B and k .

Using recent results summarised in the appendix, solvability of (6) and (8) can also be established in L^p spaces. Combining Theorem 2 and Theorem A.2 we obtain

Theorem 3 *For all $f \in B$ the integral operator $I - K_f : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$, $p \in [1, \infty]$, is bijective, with*

$$\sup_{\substack{f \in B \\ 1 \leq p \leq +\infty}} \|(I - K_f)^{-1}\|_{L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})} < \infty.$$

3. The Finite Section Method. To solve equation (11) numerically we first apply a *finite section method*, truncating the range of integration. The integral equation to solve becomes

$$(I - K_f^A)\tilde{\psi}^A = 2\tilde{g}, \quad (13)$$

for some $A > 0$, where

$$K_f^A \phi(s) = \int_{-A}^A k_f(s, t) \phi(t) dt, \quad s \in \mathbb{R}. \quad (14)$$

To proceed with the analysis we make the following assumption that the uniform bound in Theorem 2 extends also to the finite section operator (14). Of course this assumption implies that (13) is solvable and that the finite section method is stable, i.e. $\|\tilde{\psi}^A\|_\infty = O(1)$ as $A \rightarrow \infty$.

Assumption 1 *For some $A_0 > 0$, $I - K_f^A : BC(\mathbb{R}) \rightarrow BC(\mathbb{R})$ is bijective for all $A \geq A_0$, $f \in B$ and*

$$\sup_{\substack{f \in B \\ A \geq A_0}} \|(I - K_f^A)^{-1}\| < \infty.$$

From standard results on the finite section method for convolution operators, Assumption 1 holds in the special case when the boundary is flat ($c_1 = c_2$). Since (cf [3]) $\|K_f^A - K_g^A\| \rightarrow 0$ as $\|f - g\|_{C^{1,1}(\mathbb{R})} \rightarrow 0$, uniformly for $f, g \in B$, $A > 0$, it follows by operator perturbation arguments that Assumption 1 also holds for all sufficiently small $c_2 > c_1$, so that the finite section method is well-defined and stable at least for rough surfaces of sufficiently small slope. In the general case of rough surfaces of arbitrary surface amplitude and slope, it is not clear that Assumption 1 necessarily holds. However we can prove the stability of a modified finite section method in which the boundary f is flattened in finite neighborhoods of $\pm A$ — see [19] (in this proceedings) for details.

If Assumption 1 holds it follows [19] that $\tilde{\psi}^A(s) \rightarrow \tilde{\psi}(s)$, uniformly on finite intervals, and hence that $u^A(x) \rightarrow u(x)$ uniformly on bounded subsets of D , where u^A is the approximation given by the right hand side of (4) with ψ replaced by ψ^A , defined by $\psi^A((s, f(s))) := \psi^A(s)$, $|s| \leq A$, $:= 0$, $|s| > A$.

In [6] a related approximation procedure is considered, in which the rough surface is approximated by a periodic surface (diffraction grating) of period $2A$. For this procedure stability follows automatically from Theorem 2 and convergence is shown in [6].

Inspired by the *banded matrix iterative algorithm* for the integral equation (1) in [26], we will solve equation (13) iteratively, approximating K_f^A by K_f^{A,A^*} , for some $A^* \in (0, 2A]$, where

$$K_f^{A,A^*} \phi(s) := \int_{[-A,A] \cap [s-A^*, s+A^*]} k_f(s, t) \phi(t) dt, \quad s \in \mathbb{R}. \quad (15)$$

The iterative scheme, in operator form, is

$$\tilde{\psi}^{(m+1)} = (I - K_f^{A,A^*})^{-1}((K_f^A - K_f^{A,A^*})\tilde{\psi}^{(m)} + 2\tilde{g}), \quad m = 0, 1, 2, \dots, \quad (16)$$

with $\tilde{\psi}^{(0)} = 2(I - K_f^{A,A^*})^{-1}\tilde{g}$. The following theorem follows easily from the bound (5).

Theorem 1 *For all $A > 0$ and $0 < A^* \leq 2A$,*

$$\|K_f^A - K_f^{A,A^*}\| \leq CA^{*-\frac{1}{2}},$$

for every $f \in B$, where the constant $C > 0$ only depends on B and k .

Combining Theorem 1 with standard operator perturbation arguments we obtain convergence of the iterative scheme, uniformly in A , for all sufficiently large A^* .

Corollary 2 *If Assumption 1 holds then, for some $C, c, A_0^* > 0$ dependent only on B and k and all $A^* \geq A_0^*$,*

$$\|\tilde{\psi}^A - \tilde{\psi}^{(m)}\|_\infty \leq c(CA^*)^{-\frac{m+1}{2}}, \quad m = 0, 1, 2, \dots$$

4. Numerical Solution and Results. Meier *et al.* [18] propose a Nyström method for approximation of the integral operator K_f , which approximates $K_f\phi$ by a sum over values of ϕ on the uniform grid $\{jh | j \in \mathbb{Z}\}$ where $h > 0$ is the step-length. This method is superalgebraically convergent if the boundary is smooth, i.e. the error is $O(h^n)$ as $h \rightarrow 0$ for every $n > 0$. We apply this Nyström method to approximate K_f^A and K_f^{A,A^*} . Further, to reduce storage and computation time, we apply a matrix compression suggested by the canonical grid method of Tsang *et al.* [26], approximating the discretised operator $K_f^A - K_f^{A,A^*}$ by a sum of products of diagonal and Toeplitz matrices, so that a matrix-vector multiply can be carried out, using the FFT, in $O(N \log N)$ operations where $N = A/h$ so that $2N + 1$ is the number of unknowns. For details see [17].

We finish the paper by showing some numerical results. The first is for the (very simple) test case of scattering of an incident cylindrical wave $u^i(x) := \Phi(x, x_0)$, with $x_0 \in D$, by a

completely flat surface. We take $f(s) \equiv 1$ so that the scattered field is $u(x) = -\Phi(x, x_0^*)$, where x_0^* is the image of x_0 in the straight line Γ . The wavelength, $\lambda = 2\pi/k = 1$. Figure 1 plots the error in the computation of $u(x)$ on the line $x_2 = 2$ when ψ is approximated in the layer potential representation (4) by ψ_{approx} , where

$$\psi_{approx}((s, f(s))) := \begin{cases} \tilde{\psi}_h^A(s), & |s| \leq A, \\ 0, & |s| > A. \end{cases}$$

and $\tilde{\psi}_h^A$ denotes the Nyström method approximation to $\tilde{\psi}^A$. The lower curve in Figure 1 are the errors for the numerical scheme which has just been described. The upper curve (with significantly greater errors) is obtained if the identical method is used but with the half-space Dirichlet Green's function $G(x, y)$ replaced by the standard free space Green's function $\Phi(x, y)$. For both computations $h = 0.1\lambda$, $A^*/\lambda = 3$ and $A = 40\lambda$ and the iteration (16) is continued until the residual $\|\tilde{\psi}^{(m)} - K_f^A \tilde{\psi}^{(m)} - 2\tilde{g}\|_\infty \leq 10^{-12} \|2\tilde{g}\|_\infty$. Not only does use of the half-plane Dirichlet Green's function produce more accurate predictions than the free field Green's function but also, in the iterative scheme (16), far fewer iterations are required (26 for $G(x, y)$ compared to 107 for $\Phi(x, y)$).

Figure 1: Error in computed scattered field on the line $x_2 = 2\lambda$ for a flat scattering surface. Upper curve: free field Green's function; lower curve: Dirichlet half-plane Green's function.

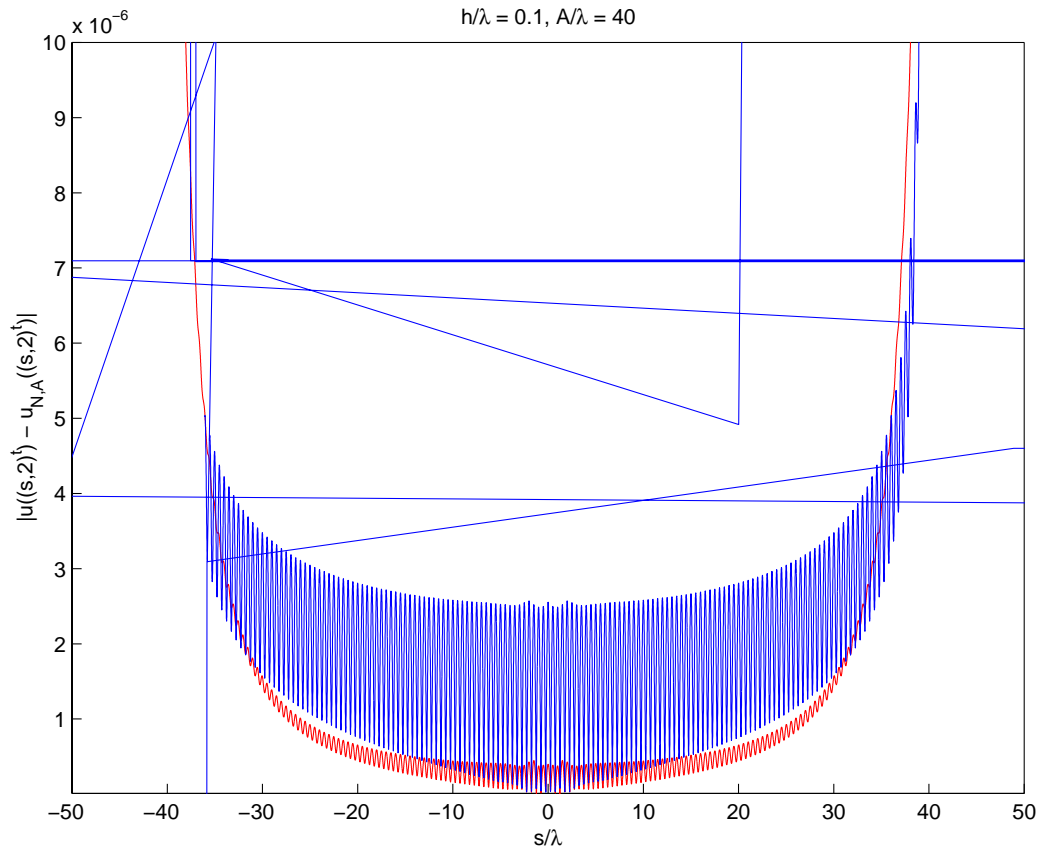


Table 1: Maximum errors in computed far field pattern for a sinusoidal surface profile.

	Dirichlet Green's Function			Free Space Green's function		
A/λ	Max. Error, e_A	EOC	No. of Iterations	Max. Error, e_A	EOC	No. of Iterations
10	0.0168		13	0.0171		17
		0.55			0.50	
20	0.0115		16	0.0121		30
		0.62				

also possible to obtain stability and convergence results for the effect of truncation of the infinite boundary surface and to prove convergence for a banded matrix iterative algorithm, similar to that of Tsang et al. 1995.

Numerical results suggest a reduction in errors introduced due to the truncation of the boundary, for the integral equation using the half-plane Green's function $G(x, y)$. They also confirm the theoretically predicted convergence of the banded matrix iterative algorithm, uniformly with respect to the length $2A$ of the finite truncated boundary arc, when the Green's function $G(x, y)$ is used.

Appendix: Solvability results for a class of integral equations on the real line. For $\kappa \in L^1(\mathbb{R})$, $\lambda \in BC(\tilde{\mathbb{R}}^2)$, where $\tilde{\mathbb{R}}^2 = \{x = (s, t) \in \mathbb{R}^2 : s \neq 0\}$, consider the integral operator K_λ defined by

$$K_\lambda \phi(s) := \int_{-\infty}^{\infty} \kappa(s-t) \lambda(s-t, t) \phi(t) dt, \quad s \in \mathbb{R}. \quad (19)$$

K_λ maps $L^\infty(\mathbb{R})$ into $BC(\mathbb{R})$, and also, by Young's theorem, $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$ for all $p \in [1, \infty)$.

Considering K_λ as an operator on $L^p(\mathbb{R})$, $p \in [1, \infty)$, the corresponding dual operator in $L^q(\mathbb{R}) = (L^p(\mathbb{R}))^*$ ($1/p + 1/q = 1$) is given by

$$(K_\lambda)^* \phi(s) = \tilde{K}_{\tilde{\lambda}} \phi(s) := \int_{-\infty}^{\infty} \tilde{\kappa}(s-t) \tilde{\lambda}(s-t, t) \phi(t) dt, \quad (20)$$

where $\tilde{\kappa}(s) := \overline{\kappa(-s)}$ and $\tilde{\lambda}(s, t) := \overline{\lambda(-s, s+t)}$. Conversely, $K_\lambda = (\tilde{K}_{\tilde{\lambda}})^*$ for $q \in [1, \infty)$.

For $(\lambda_n) \subset BC(\tilde{\mathbb{R}}^2)$ and $\lambda \in BC(\tilde{\mathbb{R}}^2)$, we say that λ_n is *s-convergent* to λ and write $\lambda_n \xrightarrow{s} \lambda$ if $\sup_n \|\lambda_n\|_{L^\infty(\mathbb{R}^2)} < \infty$ and $\lambda_n(x) \rightarrow \lambda(x)$ uniformly on every compact subset of $\tilde{\mathbb{R}}^2$. We will also call a set $\Lambda \subset BC(\tilde{\mathbb{R}}^2)$ *s-sequentially compact* if every sequence $(\lambda_n) \subset \Lambda$ has a subsequence that is s-convergent to some $\lambda \in \Lambda$. Further, introduce the translation operator $T_a^{(2)}$, defined on $BC(\tilde{\mathbb{R}}^2)$ by

$$T_a^{(2)} \lambda(s, t) = \lambda(s, t - a), \quad (s, t) \in \tilde{\mathbb{R}}^2, a \in \mathbb{R}.$$

The following theorem, proved in [23] and, as a corollary of a more general result in [6], then yields conditions for the invertibility of the integral operators $I - K_\lambda$ and $I - \tilde{K}_{\tilde{\lambda}}$.

Theorem A.1 *Assume that $\Lambda \subset BC(\tilde{\mathbb{R}}^2)$ is s-sequentially compact and satisfies that $T_a^{(2)}(\Lambda) = \Lambda$ for some $a \in \mathbb{R}$ with $a \neq 0$, and that $I - K_\lambda$ is injective on $BC(\mathbb{R})$ for all $\lambda \in \Lambda$. Then $(I - K_\lambda)^{-1}$ exists as an operator on the range space $(I - K_\lambda)(BC(\mathbb{R}))$ for all $\lambda \in \Lambda$ and*

$$\sup_{\lambda \in \Lambda} \|(I - K_\lambda)^{-1}\|_{BC(\mathbb{R}) \rightarrow BC(\mathbb{R})} < \infty.$$

Moreover, if for every $\lambda \in \Lambda$ there exists a sequence $(\lambda_n) \subset \Lambda$ such that $\lambda_n \xrightarrow{s} \lambda$ and $I - K_{\lambda_n}$ is bijective on $BC(\mathbb{R})$, then also $I - K_\lambda$ is bijective for each $\lambda \in \Lambda$.

Assume now that for some set Λ satisfying the assumptions of the previous theorem, we know that $I - K_\lambda$ is injective on $BC(\mathbb{R})$ for all $\lambda \in \Lambda$. Then, by the previous theorem we conclude that $I - K_\lambda$ is in fact bijective. In the following we will assume additionally that, for some $q > 1$,

$$\kappa(s) = O(|s|^{-q}) \quad \text{as } s \rightarrow \infty.$$

Then [1] K_λ is also well defined as a bounded operator on the weighted space

$$Y_p := \{\phi \in BC(\mathbb{R}) : \|\phi\|_{Y_p} := \sup_{s \in \mathbb{R}} |(1 + |s|)^p \phi(s)| < \infty\},$$

for $0 \leq p \leq q$. More significantly, it is shown in [11] that if $I - K_\lambda$ is bijective as an operator on $BC(\mathbb{R})$ then $I - K_\lambda$ is also bijective as an operator on Y_p and that the norms of the inverse operators are uniformly bounded for $\lambda \in \Lambda$. For $p > 1$, $Y_p \subset L^1(\mathbb{R})$ and thus $(I - K_\lambda)(L^1(\mathbb{R})) \supset (I - K_\lambda)(Y_p) = Y_p$ which is dense in $L^1(\mathbb{R})$. By a duality argument, we conclude that $(I - K_\lambda)^* = I - \tilde{K}_\lambda$ is injective on $L^\infty(\mathbb{R})$, hence on $BC(\mathbb{R})$, and thus, by a second application of Theorem A.1, bijective for all $\lambda \in \Lambda$.

Since $K_\lambda, \tilde{K}_\lambda : L^\infty(\mathbb{R}) \rightarrow BC(\mathbb{R})$ and are bounded, bijectivity of $I - K_\lambda$ and $I - \tilde{K}_\lambda$ as operators on $L^\infty(\mathbb{R})$ (and boundedness of the inverse operators uniformly in λ) is a simple consequence of bijectivity on $BC(\mathbb{R})$. By applications of duality results, Theorems 4.12 and 4.14 in Rudin (1991), it follows that both operators are bijective as operators on $L^1(\mathbb{R})$. Uniform boundedness of the inverse operators on $L_1(\mathbb{R})$ is also seen to hold. Thus, the following theorem follows by an application of the theorem of Riesz-Thorin (Stein and Weiss 1971, Chapter V):

Theorem A.2 *Assume that the assumptions of Theorem A.1 are satisfied. Then, for any $p \in [1, \infty]$, the operators $I - K_\lambda$ are bijective on $L^p(\mathbb{R})$ for all $\lambda \in \Lambda$ and*

$$\sup_{\substack{\lambda \in \Lambda \\ p \in [1, \infty]}} \|(I - K_\lambda)^{-1}\|_{L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})} < \infty.$$

References

- [1] Chandler-Wilde, S.N. 1994 On asymptotic behaviour at infinity and the finite section method for integral equations on the half-line. *J. Int. Equ. Appl.* **6**, 37–74.
- [2] Chandler-Wilde, S.N. 1997 The impedance boundary value problem for the Helmholtz equation in a half-plane. *Math. Methods Appl. Sci.* **20**, 813–840.
- [3] Chandler-Wilde, S.N. & Ross, C.R. 1996 Scattering by rough surfaces: The Dirichlet problem for the Helmholtz equation in a non-locally perturbed half-plane. *Math. Methods Appl. Sci.* **19**, 959–976.
- [4] Chandler-Wilde, S.N. & Zhang, B. 1998 A uniqueness result for scattering by infinite rough surfaces. *SIAM J. Appl. Math.* **58**, 1774–1790.
- [5] Chandler-Wilde, S.N., Ross, C.R. & Zhang, B. 1999 Scattering by infinite one-dimensional rough surfaces. *Proc. R. Soc. London, Ser. A* **455**, 3767–3787.
- [6] Chandler-Wilde, S.N., Zhang, B. & Ross, C.R. 2000 On the Solvability of Second Kind Integral Equations on the Real Line. *J. Math. Ana. Appl.*, accepted for publication.
- [7] Colton, D. & Kress, R. 1983 *Integral Equation Methods in Scattering Theory*. New York: Wiley.
- [8] DeSanto, J.A. & Brown, G.S. 1986 Analytical techniques for multiple scattering from rough surfaces. In: *Progress in Optics XXIII* (E. Wolf, ed.), 1–62. Amsterdam: Elsevier.
- [9] DeSanto, J.A., Erdmann, G., Hereman, W. & Misa, M. 1998 Theoretical and computational aspects of scattering from rough surfaces: one-dimensional perfectly reflecting surfaces. *Waves in Random Media* **8**, 385–414.
- [10] DeSanto, J.A. & Martin, P.A. 1997 On the derivation of boundary integral equations for scattering by an infinite one-dimensional rough surface. *J. Acoust. Soc. Amer.* **102**, 67–77.
- [11] Haseloh, K.O. 2000 *Lösbarkeit Fredholmscher Integralgleichungen zweiter Art in gewichteten Räumen*, Diplom thesis, Universität Hannover.
- [12] Kapp, D.A. & Brown, G.S. 1996 A new numerical method for rough surface scattering calculations. *IEEE Trans. Anten. Propag.* **44**, 711–721.
- [13] Kress, R. 1998 *Linear Integral Equations, 2nd ed.*. Berlin: Springer.
- [14] Macaskill, C. & Cao, P. 1996 A new treatment of rough surface scattering. *Proc. R. Soc. Lond.* **A 452**, 2593–2612.

- [15] Maystre, D., Saillard, M. & Ingers, J. 1991 Scattering by one- or two-dimensional randomly rough surfaces. *Waves Random Media* **3**, 143–155.
- [16] Meier, A. 1997 MSc Dissertation, Dept. Maths and Stats, Brunel University, UK.
- [17] Meier, A., Arens, T., & Chandler-Wilde, S.N. 1999a A Fast Nyström Method for Scattering by One-Dimensional Rough Surfaces, Proceedings of *Second UK Conference on Boundary Integral Methods*, Brunel University Press, 91–100.
- [18] Meier, A., Arens, T., & Chandler-Wilde, S.N. 1999b A Nyström Method for a Class of Integral Equations on the Real Line with Application to Scattering by Diffraction Gratings and Rough Surfaces, submitted for publication.
- [19] Meier, A., Chandler-Wilde, S.N. 2000 *On the stability and convergence of the finite section method for integral equation formulations of rough surface scattering*, in Proceedings of the Fifth International Conference on Mathematical and Numerical Aspects of Wave Propagation, Santiago de Compostela, Spain, July 2000.
- [20] Milder, D.M. 1996 Role of the admittance operator in rough-surface scattering. *J. Acoust. Soc. Am.* **100**, 759–768.
- [21] Ogilvy, J.A. 1991 *Theory of Wave Scattering from Random Rough Surfaces*. Bristol: Adam Hilger.
- [22] Petit, R. 1980 *Electromagnetic Theory of Gratings*. Berlin: Springer.
- [23] Ross, C.R. 1996 Direct and Inverse Scattering by Rough Surfaces, PhD Thesis, Dept. Maths and Stats, Brunel University, UK.
- [24] Rudin, W. 1991 *Functional Analysis*. New York: McGraw-Hill.
- [25] Stein, E.M., & Weiss, G. 1971 *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton University Press.
- [26] Tsang, L., Chan, C.H., Pak, K., & Sangani, H. 1995 Monte-Carlo simulations of large-scale problems of random rough surface scattering and applications to grazing incidence with the BMIA/canonical grid method. *IEEE Trans. Ant. Prop.* **43**, 851–859.
- [27] Voronovich, A.G. 1994 *Wave Scattering from Rough Surfaces*. Berlin: Springer.
- [28] Wagner, R.L., Song, J.M., & Chew, W.C. 1997 Monte-Carlo simulation of electromagnetic scattering from two-dimensional random rough surfaces. *IEEE Trans. Ant. Prop.* **45**, 235–246.
- [29] Zhang, B. & Chandler-Wilde, S.N. 1999 Integral equation methods for scattering by infinite rough surfaces, in preparation.