# Reduced Basis Output Bounds for Harmonic Wave Propagation Problems

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# Outline of the Presentation

1. Introduction
   - Presentation of the Problem
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   - The Truth Approximation
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# Outline of the Presentation

# Presentation of the Problem

## Problem

Given $\mu \in \mathcal{D} \subset \mathbb{R}^P$ we want to evaluate the output

$$s^e(\mu) \;=\; l^e(u^e(\mu);\mu) \;\in\; \mathbb{C},$$

where $u^e(\mu) \in X^e$ is the solution of the PDE

$$a^e(u^e(\mu), v; \mu) \;=\; f^e(v;\mu), \;\; v \in \; X^e,$$

with a fast procedure.

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with a fast procedure.

## Difficulty

Implicit relation between the input and the output through the resolution of a PDE.

# Application Examples

**The applicative context**

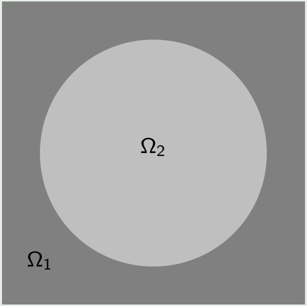
| Optimization | Control |
| --- | --- |
| $\Downarrow$ | $\Downarrow$ |
| Many queries are needed. | Real-time evaluations. |

# Application examples

## Example 1: Design problems using materials.

Application examples: photonics and microoptics,

embedded antenna.




$$\mu \equiv (\varepsilon_{|\Omega_1}, \mu_{|\Omega_1}, \varepsilon_{|\Omega_2}, \mu_{|\Omega_2}) \in \mathbb{R}^4,$$

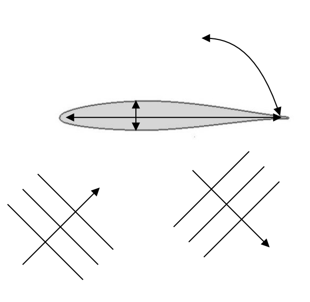
$$u(\mu) \equiv (E_x(\mu), E_y(\mu), H_z(\mu)),$$

$$s(\mu) \equiv \text{Focusing in a certain region.}$$

# Application examples

## Example 2: Active real-time control

Application examples: Noise control.



$$\mu \equiv (r_x, r_y, \phi) \in \mathbb{R}^3,$$

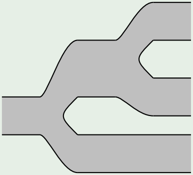
$$u(\mu) \equiv (u(\mu), v(\mu), p(\mu)),$$

$s(\mu) \equiv$ Energy on a certain part of the domain

# Application examples

## Example 3: Geometric Design using Reduced Element Method

Application examples: Modeling of hierarchic problems,
geometric design, e.g.,
minimization of scattering.



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Y. Maday, E. M. Rønquist. *A Reduced-Basis Element Method.*

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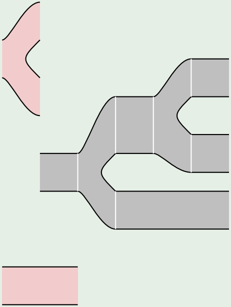

$$\mu \equiv \text{characteristic lenghts}$$

$$u(\mu) \equiv (E_x(\mu), E_y(\mu), H_z(\mu)),$$

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# Outline of the Presentation

# Obtaining an Approximate Solution

**The Goal**

To build a method providing a rapid, accurate and reliable approximation of the output.

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## The Selected Approach

1. Discretize the EDP using a (very accurate) Galerkin approximation called the truth approximation. Compute the output using this solution.
2. Apply a model order reduction method to drastically reduce the dimension of the discrete space:

Reduced basis method
- Similar accuracy.
- Much faster.
- A posteriori error estimators.

# The Truth Approximation

**Definition (Truth Approximation)**

Let us introduce the approximation space $X$ with $\dim(X) = \mathcal{N}$. The solution of the approximate problem

$$\left| \begin{array}{l} \text{Find } u(\mu) \in X \text{ such that} \\ a(u(\mu), v; \mu) \;=\; f(v), \quad v \in X, \end{array} \right.$$

is called the truth approximation of the problem.

We define the truth approximation of the output by

$$s(\mu) \;=\; f(u(\mu)).$$

# The Truth Approximation

**Remark**

When the truth approximation is non-conforming, the operators $a^e(\cdot, \cdot; \cdot), f^e(\cdot)$ and the norm $\| \cdot \|_{X^e}$ might not be well defined for elements on $X$. We'll consider discrete versions.

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## Some Requirements on the Truth Approximation

- The approximation is stable and accurate when $\mathcal{N} \mapsto \infty$.
- We assume that $\|u^e(\mu) - u(\mu)\|_X$ is suitably small.
- Consequently, $\mathcal{N}$ might be very large.

# The Truth Approximation

## Assumptions on the Operators

- $f(\cdot)$ continuous linear form:

$$\sup_{v \in X} \frac{|f(v)|}{\|v\|_X} \leq \gamma_f .$$

- $a(\cdot, \cdot; \mu)$ bilinear operator uniformly continuous on $\mathcal{D}$:

$$\gamma(\mu) := \sup_{u \in X} \sup_{v \in X} \frac{|a(u, v; \mu)|}{\|u\|_X \|v\|_X} \leq \gamma_a, \qquad \forall \mu \in \mathcal{D}.$$

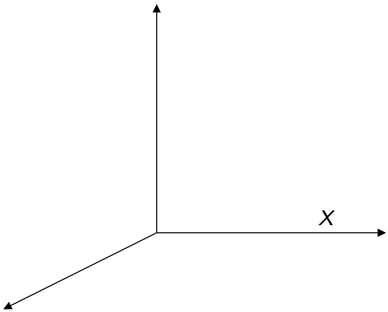
- The inf-sup parameter $\beta(\mu)$ is bounded away from zero uniformly on $\mu \in \mathcal{D}$ (well-posedness):

$$0 < \beta_0 \leq \beta(\mu) := \inf_{u \in X} \sup_{v \in X} \frac{|a(u, v; \mu)|}{\|u\|_X \|v\|_X}, \quad \forall \mu \in \mathcal{D}.$$

# The Reduced Basis Approach: A Motivation

## Observation

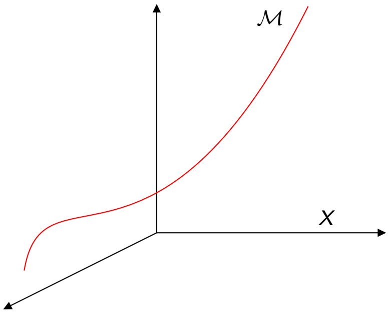
- For each $\mu$ we seek the solution on $X$.



# The Reduced Basis Approach: A Motivation

## Observation

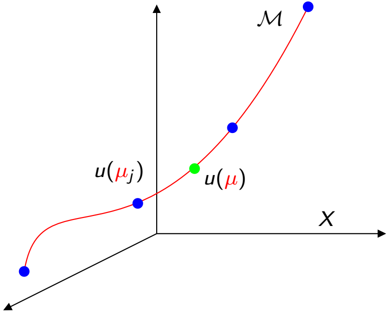
- For each $\mu$ we seek the solution on $X$.
- The set $\mathcal{M} = \{u(\mu),\ \mu \in \mathcal{D}\} \subset X$ is a smooth manifold of dimension "much smaller" than $\mathcal{N}$.



# The Reduced Basis Approach: A Motivation

## The Main Idea: Reduce the Basis

We might expect a good approximation using a Galerkin approach using solutions for "well chosen" sampling of parameters as base functions.



# The Reduced Basis Method: The Algorithm

## Step 1: Construction of the Reduced Basis.

- We introduce the nested set of samples:

$$\mathcal{S}_N = \{\mu_j \in \mathcal{D}, \quad j \in \{1, ..., N\}\}, \quad N \in \{1, \ldots, N_{MAX}\}.$$

- We construct the reduced basis approximation spaces

$$X_N = \text{span}\,(\xi_j = u(\mu_j), \quad j \in \{1, ..., N\}).$$

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## Step 2: The RB Solution. . .

- For a given $\mu \in \mathcal{D}$ we solve the RB problem

$$\left|\begin{array}{l} \text{Find } u_N(\mu) \in X_N \text{ such that} \\ a(u_N(\mu), v; \mu) = f(v), \quad \forall v \in X_N. \end{array}\right.$$

# The Reduced Basis Method: The Algorithm

**Step 2: ...and the RB Output.**

- Compute the reduced basis output

$$s_N(\mu) = f(u_N(\mu)).$$

## The Reduced Basis Method: The Algorithm

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### Remark on the Output.

We have that

$$\begin{aligned} s_N(\mu) - s(\mu) &= f(u_N(\mu) - u(\mu)) \\ &= a(u(\mu), u_N(\mu) - u(\mu)) - \overbrace{a(u_N(\mu) - u(\mu), u_N(\mu))}^{=0, \text{ Galerkin orth.}} \\ &= a(u_N(\mu) - u(\mu), u_N(\mu) - u(\mu)), \end{aligned}$$

that implies superconvergence[a].

[a] N. A. Pierce et al. *Adjoint Recovery of Superconvergent Functionals...*

# The Reduced Basis Method: Some Nice Properties...

## Properties of the Reduced Basis Method

- Exponential convergence towards the truth approximation is numerically observed for a good choice of $\mathcal{S}_N$:

$$\|u_N(\mu) - u(\mu)\|_X \approx \mathcal{C}\ e^{-\alpha N}, \qquad \alpha > 0.$$

- For many problems we can choose $N \ll \mathcal{N}$.
- A posteriori error estimates with respect to the truth approximation can be obtained, certifying the RB solution.

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## The Ultimate Goal is ...

- ... to compute $s_N(\mu)$ with a computational cost $\mathcal{N}$ independent.
- Not always possible!!! $a(w, v; \mu) = \int_\Omega g(x, \mu) w(x) v(x)\, \mathrm{d}x$

# The Reduced Basis Method: An Important Assumption

## The Affine Assumption.

If $a(\cdot,\cdot;\cdot)$ satisfies the affine assumption[a] (variable separation property)

$$a(u,v;\mu) \;=\; \sum_{q=1}^{Q_a} \Theta_q(\mu)\, a_q(u,v),$$

an off-line on-line strategy can be followed.

- The computations on the off-line part are $\mu$ independent. They are done once and for all.
- The number of computations on the on-line part is $\mathcal{N}$ independent.

[a] Barrault, Maday et al. *An "empirical interpolation" method...*

# The Reduced Basis Method: The Off-line On-line Strategy

## Computation of the Output

We have that $u_N(\mu) = \sum_{j=1}^{N} u_N^j(\mu)\ \xi_j$.

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## Computation of the RB Solution

The coefficients $u_N^j(\mu),\ i \in \{1, \ldots, N\}$ satisfy $\forall i \in \{1, \ldots, N\}$,

$$a\left(\sum_{j=1}^{N} u_N^j(\mu)\xi_j, \xi_i; \mu\right) \quad = \quad f(\xi_i),$$

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## Computation of the RB Solution

▸ Details

The coefficients $u_N^j(\mu)$, $i \in \{1, \ldots, N\}$ satisfy $\forall i \in \{1, \ldots, N\}$,

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$$\Updownarrow$$

$$\boxed{\sum_{j=1}^{N} \left(\sum_{q=1}^{Q_a} \Theta_q(\mu) \boxed{a_q(\xi_j, \xi_i)}\right)} u_N^j(\mu) \quad = \quad \boxed{\boxed{f(\xi_i)}}.$$

# References on Reduced Basis

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  - Introduced for nonlinear structural analysis:
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# References on Reduced Basis

- A global approach:
  - Efficiency greatly improved:
    - Balmes (1996) Mech. Syst. Signal. Proces. 10.
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    - Sen (2007) PhD MIT.

# Outline of the Presentation

# A Priori Theory: Well-Posedness

▸ Proof

## Theorem (Well Posedness)

*Let us assume that the discrete inf-sup parameters satisfy*

$$0 < \tilde{\beta}_0 \leq \inf_{u \in X_N} \sup_{v \in X_N} \frac{|a(u, v; \mu)|}{\|u\|_X \|v\|_X}.$$

*Then the RB problem*

$$\left| \begin{array}{l} \text{Find } u_N(\mu) \in X_N \text{ such that} \\ a(u_N(\mu), v; \mu) \; = \; f(v), \quad \forall\, v \in X_N, \end{array} \right.$$

*is well-posed.*

# A Priori Theory: Error Estimates

**Theorem (RB Solution)**

*Under the same assumptions of the previous theorem, we have that*

$$\|u(\mu) - u_N(\mu)\|_X \quad \leq \quad \left(1 + \frac{\gamma(\mu)}{\tilde{\beta}_0}\right) \inf_{w \in X_N} \|u(\mu) - w\|_X .$$

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## Conjecture on the Best Approximation Error

The best approximation[a] [b] error behaves like

$$\inf_{w \in X_N} \|u(\mu) - w\|_X \approx e^{-\alpha N}, \quad \alpha > 0.$$

---

[a] Y. Maday et al. *A Priori Convergence Theory for RB Approx...*
[b] Y. Maday, A. Buffa,... In preparation.

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**Theorem (Superconvergence on the Output)**

*Under the same assumptions, the error on the output satisfies*

$$|s(\mu) - s_N(\mu)| \leq \gamma(\mu) \left(1 + \frac{\gamma(\mu)}{\tilde{\beta}_0}\right)^2 \inf_{w \in X_N} \|u(\mu) - w\|_X^2 .$$

# A Posteriori Theory: Obtaining the Reliability

## Main Features

- An a posteriori error estimator that certifies the RB approx. with respect to the true approx. can be constructed.
- When the affine assumption is satisfied, the computation of this quantity can be performed using an off-line on-line strategy.

# A Posteriori Theory: The Main Ideas

## Truth approximation

$$X = \{\xi_1, \xi_2, \ldots, \xi_N, \eta_{N+1}, \cdots, \eta_{\mathcal{N}}\}$$

$$\begin{bmatrix} A^{\mu}_{1,1} & A^{\mu}_{1,2} \\ A^{\mu}_{2,1} & A^{\mu}_{2,2} \end{bmatrix} \begin{bmatrix} U^{\mu}_1 \\ U^{\mu}_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

## RB approximation

$$X_N = \{\xi_1, \xi_2, \ldots, \xi_N\}$$

$$A^{\mu}_{1,1} \, U^{\mu}_N = F_1$$

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## The equations for the error

$$\begin{bmatrix} A^{\mu}_{1,1} & A^{\mu}_{1,2} \\ A^{\mu}_{2,1} & A^{\mu}_{2,2} \end{bmatrix} \begin{bmatrix} U^{\mu}_1 - U^{\mu}_N \\ U^{\mu}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ F_2 - A^{\mu}_{2,1}\, U^{\mu}_N \end{bmatrix}}_{\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} - \begin{bmatrix} A^{\mu}_{1,1} & A^{\mu}_{1,2} \\ A^{\mu}_{2,1} & A^{\mu}_{2,2} \end{bmatrix} \begin{bmatrix} U^{\mu}_N \\ 0 \end{bmatrix}}$$

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$$\begin{bmatrix} A^{\mu}_{1,1} & A^{\mu}_{1,2} \\ A^{\mu}_{2,1} & A^{\mu}_{2,2} \end{bmatrix} \begin{bmatrix} U^{\mu}_1 \\ U^{\mu}_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$s^{\mu} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \cdot \begin{bmatrix} U^{\mu}_1 \\ U^{\mu}_2 \end{bmatrix}$$

## RB approximation

$$X_N = \{\xi_1, \xi_2, \ldots, \xi_N\}$$

$$A^{\mu}_{1,1}\, U^{\mu}_N = F_1$$

$$s^{\mu}_N = F_1 \cdot U^{\mu}_N.$$

# A Posteriori Theory: The Main Ideas

## The equations for the error

$$s^{\mu} - s_N^{\mu} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \cdot \begin{bmatrix} U_1^{\mu} - U_N^{\mu} \\ U_2^{\mu} \end{bmatrix}$$

# A Posteriori Theory: The Main Ideas

## The equations for the error

$$
\begin{aligned}
s^\mu - s_N^\mu &= \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \cdot \begin{bmatrix} U_1^\mu - U_N^\mu \\ U_2^\mu \end{bmatrix} \\
&= \begin{bmatrix} A_{1,1}^\mu & A_{1,2}^\mu \\ A_{2,1}^\mu & A_{2,2}^\mu \end{bmatrix} \begin{bmatrix} U_1^\mu \\ U_2^\mu \end{bmatrix} \cdot \begin{bmatrix} U_1^\mu - U_N^\mu \\ U_2^\mu \end{bmatrix}
\end{aligned}
$$

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$$
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s^{\mu} - s_N^{\mu} &= \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \cdot \begin{bmatrix} U_1^{\mu} - U_N^{\mu} \\ U_2^{\mu} \end{bmatrix} \\
&= \begin{bmatrix} A_{1,1}^{\mu} & A_{1,2}^{\mu} \\ A_{2,1}^{\mu} & A_{2,2}^{\mu} \end{bmatrix} \begin{bmatrix} U_1^{\mu} \\ U_2^{\mu} \end{bmatrix} \cdot \begin{bmatrix} U_1^{\mu} - U_N^{\mu} \\ U_2^{\mu} \end{bmatrix} \\
&= \begin{bmatrix} A_{1,1}^{\mu} & A_{1,2}^{\mu} \\ A_{2,1}^{\mu} & A_{2,2}^{\mu} \end{bmatrix} \begin{bmatrix} U_1^{\mu} - U_N^{\mu} \\ U_2^{\mu} \end{bmatrix} \cdot \begin{bmatrix} U_1^{\mu} - U_N^{\mu} \\ U_2^{\mu} \end{bmatrix}
\end{aligned}
$$

# A Posteriori Theory: The Main Ideas

## The equations for the error

$$
\left|\begin{array}{rl}
s^\mu - s_N^\mu & = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \cdot \begin{bmatrix} U_1^\mu - U_N^\mu \\ U_2^\mu \end{bmatrix} \\
& = \begin{bmatrix} A_{1,1}^\mu & A_{1,2}^\mu \\ A_{2,1}^\mu & A_{2,2}^\mu \end{bmatrix} \begin{bmatrix} U_1^\mu \\ U_2^\mu \end{bmatrix} \cdot \begin{bmatrix} U_1^\mu - U_N^\mu \\ U_2^\mu \end{bmatrix} \\
& = \begin{bmatrix} A_{1,1}^\mu & A_{1,2}^\mu \\ A_{2,1}^\mu & A_{2,2}^\mu \end{bmatrix} \begin{bmatrix} U_1^\mu - U_N^\mu \\ U_2^\mu \end{bmatrix} \cdot \begin{bmatrix} U_1^\mu - U_N^\mu \\ U_2^\mu \end{bmatrix} \\
= \begin{bmatrix} 0 \\ F_2 - A_{2,1}^\mu \, U_N^\mu \end{bmatrix} \cdot & \begin{bmatrix} A_{1,1}^\mu & A_{1,2}^\mu \\ A_{2,1}^\mu & A_{2,2}^\mu \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ F_2 - A_{2,1}^\mu \, U_N^\mu \end{bmatrix}
\end{array}\right.
$$

# A Posteriori Theory: Preliminaries

## One Assumption and Some Notation

- We assume that we can build $\tilde{\beta}(\mu)$ such that

$$0 \; < \; \tilde{\beta}_0 \; \leq \; \tilde{\beta}(\mu) \; \leq \; \beta(\mu), \quad \forall \; \mu \in \mathcal{D},$$

  with a (low) on-line evaluation cost independent on $\mathcal{N}$.
- We define the residual of the reduced basis problem by

$$R(v; \mu) \;\; := \;\; f(v) \; - \; a(u_N(\mu), v; \mu), \;\; \forall \; v \in X,$$

  and we introduce its dual norm

$$\varepsilon_N(\mu) \;\; := \;\; \sup_{v \in X} \frac{|R(v; \mu)|}{\|v\|_X}.$$

# A Posteriori Theory: Estimator for the RB Solution

▸ Proof

**Definition (Error Estimator for the RB Solution)**

We define the error estimator for the reduced basis solution by

$$\Delta_N(\mu) \quad := \quad \frac{\varepsilon_N(\mu)}{\tilde{\beta}(\mu)}, \qquad \forall\ \mu \in \mathcal{D}.$$

**Theorem (Efficiency)**

*The efficiency of the estimator*

$$\eta_N(\mu) \quad := \quad \frac{\Delta_N(\mu)}{\|u_N(\mu) - u(\mu)\|_X},$$

*satisfies the following inequalities*

$$1 \quad \leq \quad \eta_N(\mu) \quad \leq \quad \frac{\gamma(\mu)}{\tilde{\beta}(\mu)}, \quad \forall\ \mu \in \mathcal{D},\ \forall\ N \in \{1, \ldots, N_{MAX}\}.$$

# A Posteriori Theory: Estimator for the RB Output

Proof

**Definition (Error Estimator for the Output)**

We define the error estimator for the output by

$$\Delta_N^s(\mu) \;:=\; \frac{\varepsilon_N(\mu)^2}{\tilde{\beta}(\mu)}, \qquad \forall\; \mu \in \mathcal{D}.$$

**Theorem (Efficiency)**

*The efficiency of the estimator for the output*

$$\eta_N^s(\mu) \;:=\; \frac{\Delta_N^s(\mu)}{|s_N(\mu) \;-\; s(\mu)|},$$

*satisfies,*

$$1 \;\leq\; \eta_N^s(\mu), \quad \forall\; \mu \in \mathcal{D}, \forall\; N \in \{1, \ldots, N_{MAX}\}.$$

# A Posteriori Theory: How to Compute $\varepsilon_N(\mu)$

## Deriving an Off-line On-line Strategy...

Using the Riesz Representation Theorem we know that

$$\varepsilon_N(\mu) \quad := \quad \sup_{v \in X} \frac{|R(v;\mu)|}{\|v\|_X} \quad = \quad \|\rho^X_{R(\cdot;\mu)}\|_X,$$

where $\rho^X_{R(\cdot;\mu)} \in X$ satisfies $\forall\ v \in\ X$

$$\begin{aligned} (\rho^X_{R(\cdot;\mu)}, v)_X &= R(v;\mu) \\ &= f(v) - a(u_N(\mu), v; \mu) \end{aligned}$$

# A Posteriori Theory: How to Compute $\varepsilon_N(\mu)$

## Deriving an Off-line On-line Strategy...

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where $\rho^X_{R(\cdot;\mu)} \in X$ satisfies $\forall\ v \in\ X$

$$\begin{aligned}(\rho^X_{R(\cdot;\mu)}, v)_X \quad &= \quad R(v;\mu) \\ &= \quad f(v) - a(u_N(\mu), v; \mu) \quad = \end{aligned}$$

$$f(v) \; - \; \sum_{q=1}^{Q_a} \sum_{k=1}^{N} \Theta_q(\mu)\; u_N^k(\mu)\; a_q(\xi_k, v).$$

# A Posteriori Theory: How to Compute $\varepsilon_N(\mu)$

## Deriving an Off-line On-line Strategy... cont...

Thus, by linear superposition and uniqueness

$$\rho^X_{R(\cdot;\mu)} = \rho^X_{f(\cdot)} - \sum_{q=1}^{Q_a} \sum_{k=1}^{N} \Theta_q(\mu)\ u_N^k(\mu)\ \rho^X_{a_q(\xi_k,\cdot)},$$

and consequently

$$\varepsilon_N(\mu) = \Bigg[ (\rho^X_{f(\cdot)}, \rho^X_{f(\cdot)})_X + \sum_{q=1}^{Q_a} \sum_{k=1}^{N} \sum_{\tilde q=1}^{Q_a} \sum_{\tilde k=1}^{N} u_N^k(\mu)\ \Theta_q(\mu)\ \overline{u_N^{\tilde k}(\mu)}\ \overline{\Theta_{\tilde q}(\mu)}\ (\rho^X_{a_q(\xi_k,\cdot)}, \rho^X_{a_{\tilde q}(\xi_{\tilde k},\cdot)})_X - 2 \sum_{\tilde q=1}^{Q_a} \sum_{\tilde k=1}^{N} \Re \left[ \overline{u_N^{\tilde k}(\mu)}\ \overline{\Theta_a^{\tilde q}(\mu)}\ (\rho^X_{f(\cdot)}, \rho^X_{a^{\tilde q}(\xi_{\tilde k},\cdot)})_X \right] \Bigg]^{\frac{1}{2}}.$$

# A Posteriori Theory: How to Compute $\varepsilon_N(\mu)$

## Deriving an Off-line On-line Strategy... cont... ‣ Details

Thus, by linear superposition and uniqueness

$$\rho^X_{R(\cdot;\mu)} = \rho^X_{f(\cdot)} - \sum_{q=1}^{Q_a}\sum_{k=1}^{N} \Theta_q(\mu)\; u_N^k(\mu)\; \rho^X_{a_q(\xi_k,\cdot)},$$

and consequently

$$\varepsilon_N(\mu) = \Bigg[ (\rho^X_{f(\cdot)}, \rho^X_{f(\cdot)})_X + \sum_{q=1}^{Q_a}\sum_{k=1}^{N}\sum_{\tilde q=1}^{Q_a}\sum_{\tilde k=1}^{N} u_N^k(\mu)\; \Theta_q(\mu)\; \overline{u_N^{\tilde k}(\mu)}\; \overline{\Theta_{\tilde q}(\mu)}\; (\rho^X_{a_q(\xi_k,\cdot)}, \rho^X_{a_{\tilde q}(\xi_{\tilde k},\cdot)})_X - 2\sum_{\tilde q=1}^{Q_a}\sum_{\tilde k=1}^{N} \Re\Big[ \overline{u_N^{\tilde k}(\mu)}\; \overline{\Theta_a^{\tilde q}(\mu)}\; (\rho^X_{f(\cdot)}, \rho^X_{a^{\tilde q}(\xi_{\tilde k},\cdot)})_X \Big] \Bigg]^{\frac{1}{2}}.$$

# A Posteriori Theory: How to Compute $\tilde{\beta}(\mu)$

## Characterization of $\beta(\mu)$

$$\beta(\mu) \;:=\; \inf_{u \in X} \sup_{v \in X} \frac{|a(u, v; \mu)|}{\|u\|_X \|v\|_X}$$

# A Posteriori Theory: How to Compute $\tilde{\beta}(\mu)$

## Characterization of $\beta(\mu)$

Using the Riesz Representation Theorem we have

$$\beta(\mu) := \inf_{u \in X} \sup_{v \in X} \frac{|a(u, v; \mu)|}{\|u\|_X \|v\|_X} = \inf_{u \in X} \frac{\|\rho^X_{a(u,\cdot;\mu)}\|_X}{\|u\|_X}.$$

# A Posteriori Theory: How to Compute $\tilde{\beta}(\mu)$

## Characterization of $\beta(\mu)$

Go to $\tau(\mu, \tilde{\mu})$

Using the Riesz Representation Theorem we have

$$\beta(\mu) := \inf_{u \in X} \sup_{v \in X} \frac{|a(u, v; \mu)|}{\|u\|_X \|v\|_X} = \sqrt{\inf_{u \in X} \frac{b(u, u; \mu)}{(u, u)_X}},$$

where

$$b(u, v; \mu) = (\rho^X_{a(u,\cdot;\mu)}, \rho^X_{a(v,\cdot;\mu)})_X, \qquad \forall\, (u, v) \in X^2.$$

# A Posteriori Theory: How to Compute $\tilde{\beta}(\mu)$

## Characterization of $\beta(\mu)$

Using the Riesz Representation Theorem we have

$$\beta(\mu) := \inf_{u \in X} \sup_{v \in X} \frac{|a(u,v;\mu)|}{\|u\|_X \|v\|_X} = \sqrt{\inf_{u \in X} \frac{b(u,u;\mu)}{(u,u)_X}},$$

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$$b(u,v;\mu) = (\rho^X_{a(u,\cdot;\mu)}, \rho^X_{a(v,\cdot;\mu)})_X, \qquad \forall\, (u,v) \in X^2.$$

We introduce the hermitian generalized eigenvalue problem:

$$\left| \begin{array}{l} \text{Find } (\theta(\mu), \lambda(\mu)) \in X \times \mathbb{R}^+ \\ b(\theta(\mu), v;\mu) = \lambda(\mu)(\theta(\mu), v)_X, \quad \forall\, v \in X. \end{array} \right.$$

# A Posteriori Theory: How to Compute $\tilde{\beta}(\mu)$

## Characterization of $\beta(\mu)$

Using the Riesz Representation Theorem we have

$$\beta(\mu) := \inf_{u \in X} \sup_{v \in X} \frac{|a(u, v; \mu)|}{\|u\|_X \|v\|_X} = \sqrt{\inf_{u \in X} \frac{b(u, u; \mu)}{(u, u)_X}},$$

where

$$b(u, v; \mu) = (\rho^X_{a(u,\cdot;\mu)}, \rho^X_{a(v,\cdot;\mu)})_X, \qquad \forall\, (u, v) \in X^2.$$

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Using Raleigh's quotient arguments: $\beta(\mu) = \sqrt{\lambda_{min}(\mu)}$.

# A Posteriori Theory: How to Compute $\tilde{\beta}(\mu)$

For the sake of clarity we'll assume that

$$a(u, v; \mu) \; = \; a_{P+1}(u, v) \; + \; \sum_{q=1}^{P} \mu_q \; a_q(u, v), \quad \forall \; (u, v) \in X^2.$$

# A Posteriori Theory: How to Compute $\tilde{\beta}(\mu)$

We introduce

$$\tilde{b}(u, v; \mu; \tilde{\mu}) \quad := \quad b(u, v; \tilde{\mu}) \; + \; \sum_{p=1}^{P} (\mu_p - \tilde{\mu}_p) \, \frac{\partial b}{\partial \mu_p}(u, v; \tilde{\mu}).$$

We define the set

$$\mathcal{D}_{\tilde{\mu}} \; = \; \left\{ \mu \in \mathcal{D}, \; / \; \tilde{b}(u, u; \mu; \tilde{\mu}) \; \geq \; 0, \; \forall \; u \in X \right\},$$

and

$$\tau(\mu, \tilde{\mu}) \; := \; \sqrt{\inf_{u \in X} \frac{\tilde{b}(u, u; \mu; \tilde{\mu})}{(u, u)_X}}, \quad \forall \; \mu \in \mathcal{D}_{\tilde{\mu}}.$$

Go to $\beta(\mu)$

# A Posteriori Theory: How to Compute $\tilde{\beta}(\mu)$

We introduce

$$\tilde{b}(u,v;\mu;\tilde{\mu}) \;:=\; b(u,v;\tilde{\mu}) \;+\; \textstyle\sum_{p=1}^{P}(\mu_p-\tilde{\mu}_p)\,\dfrac{\partial b}{\partial \mu_p}(u,v;\tilde{\mu}).$$

We define the set

$$\mathcal{D}_{\tilde{\mu}} \;=\; \left\{\mu\in\mathcal{D},\ /\ \tilde{b}(u,u;\mu;\tilde{\mu}) \;\geq\; 0,\ \forall\ u\in X\right\},$$

and

$$\tau(\mu,\tilde{\mu}) \;:=\; \sqrt{\inf_{u\in X}\frac{\tilde{b}(u,u;\mu;\tilde{\mu})}{(u,u)_X}},\quad \forall\ \mu\in\mathcal{D}_{\tilde{\mu}}.$$

Go to $\beta(\mu)$

## Lemma (Properties of $\mathcal{D}_{\tilde{\mu}}$ and $\tau(\mu,\tilde{\mu})$)

- *The set $\mathcal{D}_{\tilde{\mu}}$ is convex.*
- *The function $\tau(\mu,\tilde{\mu})$ is concave in $\mu$ over $\mathcal{D}_{\tilde{\mu}}$.*
- *We have that $\beta(\mu) \;\geq\; \tau(\mu,\tilde{\mu}),\quad \forall\ \mu\in\mathcal{D}_{\tilde{\mu}}$.*

# A Posteriori Theory: How to Compute $\tilde{\beta}(\mu)$

## Construction of the Lower Bound $\tilde{\beta}(\mu)$

▸ Model Problem

- We consider $\epsilon_\beta \in (0,1)$.
- We introduce a set of points $\tilde{\mu}_j, j \in \{1,\ldots,J\}$ and associated polytopes $\mathcal{P}_{\tilde{\mu}_j} \subset \mathcal{D}_{\tilde{\mu}_j}$ such that

$$\left| \begin{array}{lll} \mathcal{D} \subset \bigcup_{j=1}^{J} \mathcal{P}_{\tilde{\mu}_j}, & & \text{Coverage condition} \\ \min_{\mu \in \mathcal{P}_{\tilde{\mu}_j}} \tau(\mu;\tilde{\mu}_j) \geq \epsilon_\beta \, \beta(\tilde{\mu}_j). & & \text{Positivity condition} \end{array} \right.$$

- We define the lower bound of the inf-sup parameter by

$$\tilde{\beta}(\mu) = \min_{j \in \{1,\ldots,J\}, \mu \in \mathcal{P}_{\tilde{\mu}_j}} \epsilon_\beta \, \beta(\tilde{\mu}_j).$$

▸ Geometr. Interpr.

# On the Construction of the Reduced Basis

## Algorithm

- Construct $\mathcal{S}$ a very fine mesh of the range of parameters.
- Select $\mu_1$ randomly.
- For $j = 1, \ldots, N_{MAX}$ do
  - Compute $u(\mu_j)$ and add it to the reduced basis. Orthonormalize. The reduced basis dim. is $N = j$.
  - Update the reduced basis information.
  - Select $\mu_{j+1}$ such that

$$\mu_{j+1} = \arg\max_{\mu \in \mathcal{S}} \left[ \textbf{Dist} \left( u(\mu), u_N(\mu) \right) \right].$$

# On the Construction of the Reduced Basis

The computational cost of the algorithm will highly depend on the choice of the distance $\mathbf{Dist}(\cdot, \cdot)$.

# On the Construction of the Reduced Basis

The computational cost of the algorithm will highly depend on the choice of the distance $\mathbf{Dist}(\cdot, \cdot)$.

## About $\mathbf{Dist}(\cdot, \cdot)$

- If $\mathbf{Dist}\,(u(\mu), u_N(\mu)) \;=\; \|u(\mu) - u_N(\mu)\|_X$ it will be very expensive. It is included on the off-line part...
- If $\mathbf{Dist}\,(u(\mu), u_N(\mu)) \;=\; \Delta_N(\mu)$ the method is much cheaper. We only compute the true approximation for the $N_{MAX}$ selected parameters.

# Extensions: When $f(\cdot;\mu) \neq l(\cdot;\mu)\dots$

Given $\mu \in \mathcal{D} \subset \mathbb{R}^P$ we want to evaluate the output

$$s(\mu) \;=\; l(u(\mu);\mu) \;\in\; \mathbb{C},$$

where $u(\mu) \in X$ is the solution of the PDE

$$a(u(\mu), v; \mu) \;=\; f(v;\mu), \;\; v \in \; X.$$

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---

Obtaining the adjoint state $\psi(\mu) \in X$ such that

$$a(\phi, \psi(\mu); \mu) \;=\; l(\phi;\mu), \;\; \phi \in \; X,$$

it turns out that

$$f(\psi(\mu);\mu) \;=\; a(u(\mu), \psi(\mu); \mu) \;=\; l(u(\mu);\mu) \;=\; s(\mu).$$

# Extensions: When $f(\cdot;\mu) \neq l(\cdot;\mu)$...

## Step 1: Construction of the Primal and Dual Reduced Basis

- We introduce the nested set of samples:

$$\begin{aligned} \mathcal{S}_N^{pr} &= \left\{\mu_j^{pr} \in \mathcal{D}, \quad j \in \{1,...,N\}\right\}, \quad N \in \{1,\dots,N_{MAX}\}, \\ \mathcal{S}_M^{du} &= \left\{\mu_j^{du} \in \mathcal{D}, \quad j \in \{1,...,M\}\right\}, \quad M \in \{1,\dots,M_{MAX}\}, \end{aligned}$$

- We construct the reduced basis approximation spaces for the primal and dual problems

$$\begin{aligned} X_N^{pr} &= \text{span}\left(\xi_j^{pr} = u(\mu_j^{pr}), \quad j \in \{1,...,N\}\right), \\ X_M^{du} &= \text{span}\left(\xi_j^{du} = \psi(\mu_j^{du}), \quad j \in \{1,...,M\}\right). \end{aligned}$$

# Extensions: When $f(\cdot;\mu) \neq l(\cdot;\mu)$...

## Step 2: The Primal and Dual RB Solutions. The RB Output

- Given $\mu \in \mathcal{D}$

$$\left| \begin{array}{l} \text{Find } u_N(\mu) \in X_N^{pr} \text{ and } \psi_M(\mu) \in X_M^{du} \text{ such that} \\ \begin{aligned} a(u_N(\mu), v; \mu) &= f(v;\mu), \quad \forall v \in X_N^{pr}, \\ a(\phi, \psi_M(\mu); \mu) &= l(\phi;\mu), \quad \forall \phi \in X_M^{du}. \end{aligned} \end{array} \right.$$

- Compute the reduced basis output

$$s_N(\mu) = l(u_N(\mu);\mu).$$

$$\begin{aligned} s(\mu) - s_N(\mu) &= l(u(\mu) - u_N(\mu);\mu) \\ &= a(u(\mu) - u_N(\mu), \psi(\mu);\mu) \end{aligned}$$

# Extensions: When $f(\cdot;\mu) \neq l(\cdot;\mu)$...

## Step 2: The Primal and Dual RB Solutions. The RB Output

- Given $\mu \in \mathcal{D}$

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- Compute the reduced basis output

$$s_N(\mu) = l(u_N(\mu);\mu).$$

$$\begin{aligned} s(\mu) - s_N(\mu) &= l(u(\mu) - u_N(\mu);\mu) \\ &= a(u(\mu) - u_N(\mu), \psi(\mu);\mu) \\ &= a(u(\mu) - u_N(\mu), \psi(\mu) - \psi_M(\mu);\mu) + \\ &\quad f(\psi_M(\mu);\mu) - a(u_N(\mu), \psi_M(\mu);\mu). \end{aligned}$$

# Extensions: When $f(\cdot;\mu) \neq l(\cdot;\mu)\dots$

## Step 2: The Primal and Dual RB Solutions. The RB Output

- Given $\mu \in \mathcal{D}$

$$\left|\begin{array}{lcl} \multicolumn{3}{l}{\text{Find } u_N(\mu) \in X_N^{pr} \text{ and } \psi_M(\mu) \in X_M^{du} \text{ such that}} \\ a(u_N(\mu), v; \mu) & = & f(v;\mu), \quad \forall v \in X_N^{pr}, \\ a(\phi, \psi_M(\mu); \mu) & = & l(\phi;\mu), \quad \forall \phi \in X_M^{du}. \end{array}\right.$$

- Compute the reduced basis output

$$s_{N,M}(\mu) = l(u_N(\mu);\mu) - a(u_N(\mu), \psi_M(\mu);\mu) + f(\psi_M(\mu);\mu).$$

$$\begin{aligned} s(\mu) - s_N(\mu) &= l(u(\mu) - u_N(\mu);\mu) \\ &= a(u(\mu) - u_N(\mu), \psi(\mu);\mu) \\ &= a(u(\mu) - u_N(\mu), \psi(\mu) - \psi_M(\mu);\mu) + \\ & \quad\; f(\psi_M(\mu);\mu) - a(u_N(\mu), \psi_M(\mu);\mu). \end{aligned}$$

# The NonAffine Case

## Difficulty

When

$$a(v, w; \mu) \;=\; \int_\Omega g(x, \mu) v(x) w(x) \,\mathrm{d}x \;+\cdots$$

the off-line on-line computational strategy does not apply.

## The Main Idea

We approximate $g(x, \mu)$ by

$$g_M(x, \mu) \;=\; \sum_{m=1}^{M} \varphi_m^M(\mu) q_m(x).$$

The approximate bilinear functional will satisfy the affine assumption.

# The NonAffine Case

## Some Requirements for the Approximation

To reduce the computational cost of the on-line procedure:

- The evaluation of $\varphi_m^M(\mu)$ must have a low computational cost ($\mathcal{N}$ independent).
- $M$ must be as small as possible.

# The NonAffine Case

## The Selected Approach

We apply:

- An interpolation procedure for the computation of $\varphi_m^M(\mu)$. In consequence, we also have to determine the interpolation points and a interpolation prodedure.
- An approximation space for the expansion exploiting the $\mu$ dependence:

$$
\begin{aligned}
\mathcal{S}_M^g &= \left\{\mu_i^g, \quad i \in \{1, \ldots, M\}\right\}, \\
W_M^g &= \operatorname{span}\left(g(\cdot;\mu_i^g), \quad i \in \{1, ..., M\}\right) \\
&= \operatorname{span}\left(q_i(\cdot), \quad i \in \{1, ..., M\}\right).
\end{aligned}
$$

# The NonAffine Case

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&= \operatorname{span}\left(q_i(\cdot), \quad i \in \{1, ..., M\}\right).
\end{aligned}
$$

## Some Notation

$$
\begin{aligned}
g_M^*(\cdot, ) &= \arg\min_{z \in W_M^g} \|g(\cdot, \mu) - z\|, \\
\varepsilon_M^*(\mu) &= \|g(\cdot, \mu) - g_M^*(\cdot, \mu)\|.
\end{aligned}
$$

# The NonAffine Case

## Building the Approximation Space

Let $\Xi_g$ be a rather fine mesh of $\mathcal{D}$.

- $\mu_1^g = \arg\min_{\mu \in \Xi_g} \|g(\cdot, \mu)\|$,
- $\mathcal{S}_1^g = \{\mu_1^g\}$,
- $W_1^g = span(g(\cdot, \mu_1^g)$,
- For $M \in \{2, ..., M_{max}\}$
    - $\mu_M^g = \arg\min_{\mu \in \Xi_g} \varepsilon_{M-1}^*(\mu)$,
    - $\mathcal{S}_M^g = \mathcal{S}_{M-1}^g \cup \{\mu_M^g\}$,
    - $W_M^g = span(g(\cdot, \mu_m^g), \quad m \in \{1, ..., M\})$.
- EndFor

# The NonAffine Case

## Defining the Interpolation Points

- $x_1^g = \arg \operatorname{ess\,sup}_{x \in \Omega} |g(x, \mu_1^g)|$,
- $q_1(\cdot) = \dfrac{g(\cdot, \mu_1^g)}{g(x_1^g, \mu_1^g)}$ ,
- For $M \in \{1, \ldots, M_{max}\}$
  - We solve $\sum_{j=1}^{M-1} \sigma_j^{M-1} q_j(x_i^g) \; = \; g(x_i^g, \mu_M^g)$,
    $\{1 \leq i \leq M-1\}$
  - $r_M(\cdot) = g(\cdot, \mu_M^g) - \sum_{j=1}^{M-1} \sigma_j^{M-1} q_j(\cdot)$,
  - $x_M^g = \arg \operatorname{ess\,sup}_{x \in \Omega} \; |r_M(\cdot)|$,
  - $q_M(\cdot) = \dfrac{r_M(\cdot)}{r_M(x_M^g)}$,
- EndFor

# The NonAffine Case

## The Interpolation Procedure

In this way, for a given $\mu \in \mathcal{D}$,

$$g_M(x, \mu) = \sum_{m=1}^{M} \varphi_m^M(\mu)\; q_m(x),$$

where $\varphi_m^M(\mu)$ satisfy the lower triangular linear system

$$\sum_{m=1}^{M} B_{l,m}\, \varphi_m^M(\mu) \;=\; g(x_l^g, \mu), \qquad 1 \leq l \leq M,$$

with $B_{l,m} \;=\; q_m(x_l^g)$.

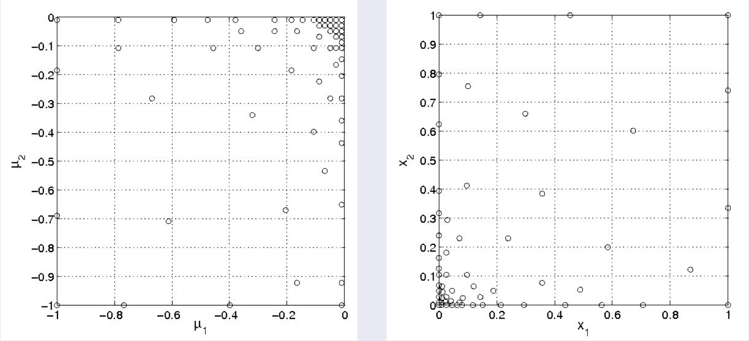
# The NonAffine Case

## Numerical Example

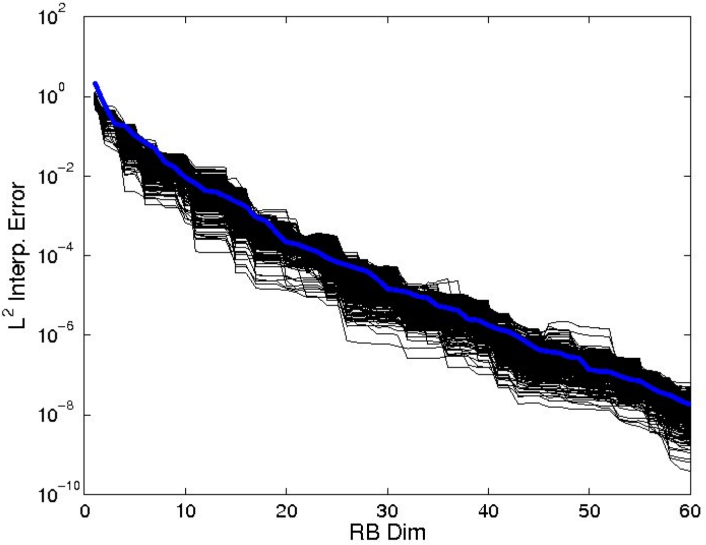
We interpolate the function

$$g(x, \mu) = \frac{1}{\sqrt{(x_1 - \mu_1)^2 + (x_2 - \mu_2)^2}}$$

where $\Omega = [0, 1]^2$ and $\mathcal{D} = [-1, -0.1]^2$.



# The NonAffine Case



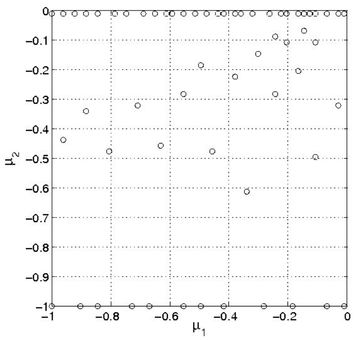
# The NonAffine Case

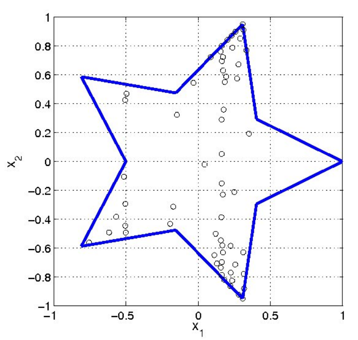
## Numerical Example

We interpolate the function

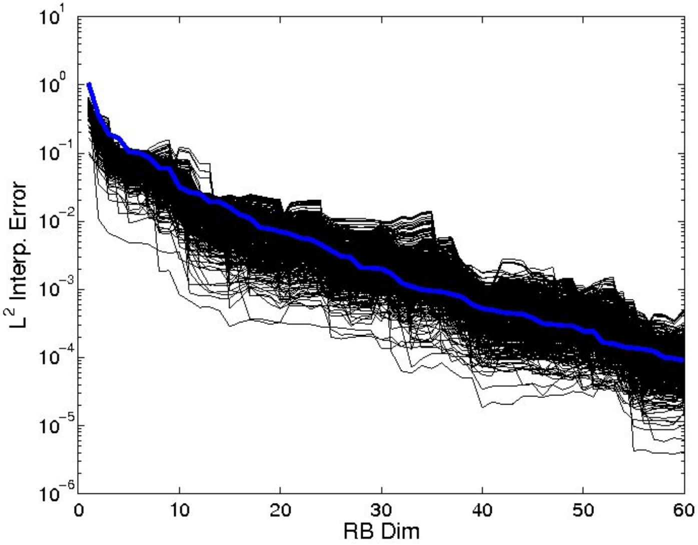
$$g(x, \mu) = \frac{1}{\sqrt{2 - \mu_1 - \mu_2 - \sin(5\pi\mu_1 x_2) - \sin(3\pi x_1)}}$$

where $\Omega = STAR\_DOMAIN$ and $\mathcal{D} = [-1, -0.1]^2$.



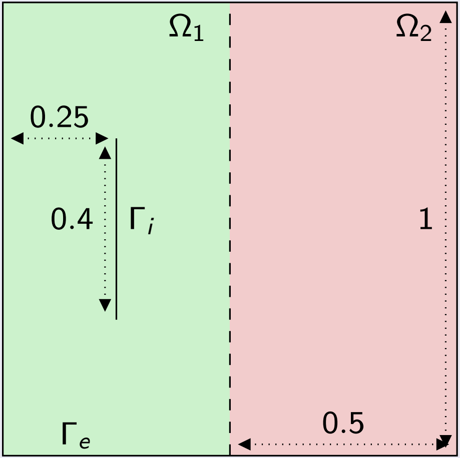
# The NonAffine Case



# Outline of the Presentation

# The Model Problem: A 2D Maxwell Application

## Geometry of the Problem



# The Model Problem: A 2D Maxwell Application

## The Primal Problem

$$\left| \begin{array}{rcll} -\omega^2 \varepsilon E_x^e & = & \dfrac{\partial}{\partial y}\left(\dfrac{1}{\mu}\dfrac{\partial E_x^e}{\partial y} - \dfrac{1}{\mu}\dfrac{\partial E_y^e}{\partial x}\right) - i\omega J_x^e, & \text{in } \Omega, \\ -\omega^2 \varepsilon E_y^e & = & -\dfrac{\partial}{\partial x}\left(\dfrac{1}{\mu}\dfrac{\partial E_x^e}{\partial y} - \dfrac{1}{\mu}\dfrac{\partial E_y^e}{\partial x}\right) - i\omega J_y^e, & \text{in } \Omega, \\ E_x^e n_y & = & E_y^e n_x, & \text{on } \Gamma_e, \end{array} \right.$$

## The RHS and Output

$$\left| \begin{array}{rcl} J_x^e & = & 0, \\ J_y^e & = & \cos\left(\omega\left(y - \dfrac{1}{2}\right)\right)\delta_{\Gamma_i}, \\ s^e((E_x, E_y)) & = & \displaystyle\int_{\Omega_2} E_x^e + E_y^e \, dx. \end{array} \right.$$

## The Coefficients

$$\left| \begin{array}{rcl} \omega & = & \dfrac{5\pi}{2}, \quad \mu = 1, \quad \text{in } \Omega, \\ \varepsilon & = & \begin{cases} \varepsilon_1 = 1, & \text{in } \Omega_1, \\ \varepsilon_2 \in [1,4], & \text{in } \Omega_2. \end{cases} \end{array} \right.$$

# The Model Problem: A 2D Maxwell Application

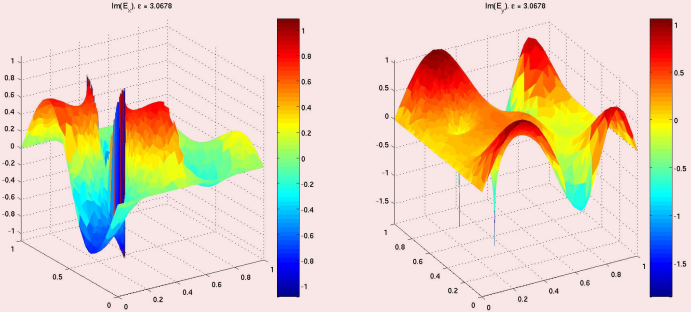
## The Dual Problem

$$
\left|
\begin{array}{rcll}
-\omega^2 \varepsilon \Psi_x^e & = & \dfrac{\partial}{\partial y}\left(\dfrac{1}{\mu}\dfrac{\partial \Psi_x^e}{\partial y} - \dfrac{1}{\mu}\dfrac{\partial \Psi_y^e}{\partial x}\right) + \chi_{\Omega_2}, & \text{in } \Omega, \\
-\omega^2 \varepsilon \Psi_y^e & = & -\dfrac{\partial}{\partial x}\left(\dfrac{1}{\mu}\dfrac{\partial \Psi_x^e}{\partial y} - \dfrac{1}{\mu}\dfrac{\partial \Psi_y^e}{\partial x}\right) + \chi_{\Omega_2}, & \text{in } \Omega, \\
\Psi_x^e n_y & = & \Psi_y^e n_x, & \text{on } \Gamma_e,
\end{array}
\right.
$$

# The Model Problem: A 2D Maxwell Application

## Difficulties of the Problem

- Singularities on the tips of the antenna.



# The Model Problem: A 2D Maxwell Application

## Difficulties of the Problem

- Singularities on the tips of the antenna.
- For several values of $\varepsilon_2$ the frequency $\omega = 5\pi/2$ is a resonance.

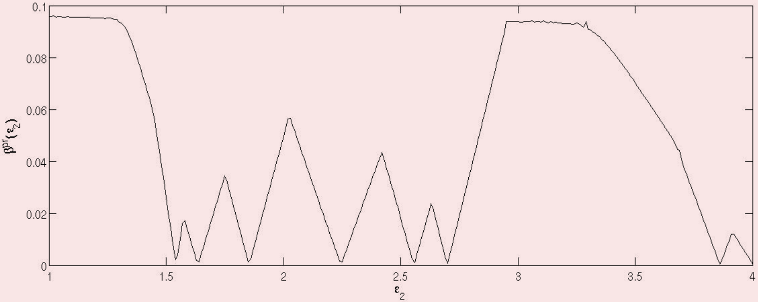
| $\varepsilon_2$ : $\omega = 5\pi/2$ is a resonance | | | |
|---|---|---|---|
| 1.5434 | 1.6357 | 1.8532 | 2.2456 |
| 2.5569 | 2.6983 | 3.8615 | 4.0033 |

For those values, $\beta(\varepsilon_2)$ vanishes!

# The Model Problem: A 2D Maxwell Application

## Difficulties of the Problem

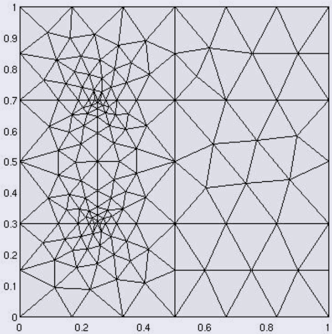
- Singularities on the tips of the antenna.
- For several values of $\varepsilon_2$ the frequency $\omega = 5\pi/2$ is a resonance.



# The True Approximation: Discontinuous Galerkin

▸ Details

- Mesh locally refined on the tips of the antenna.
- # elements = 282.
- Polynomials of order 4.
- Points/$\lambda \geq 12.5$.
- # Degrees of freedom $\mathcal{N} = 11844$.



# Outline of the Presentation

# Conclusions and Future Work

## Conclusions

- We have shown the feasibility of a RB approach for harmonic time dependent wave propagation problems using DG for the true approximation.
- We remark numerically an exponential rate of convergence on the number of elements of the basis.
- An a posteriori error estimator that certifies the RB solution with respect to the true approximation can be constructed.
- We have presented efficient methods for the basis construction.

# Conclusions and Future Work

## Conclusions

- We have shown the feasibility of a RB approach for harmonic time dependent wave propagation problems using DG for the true approximation.
- We remark numerically an exponential rate of convergence on the number of elements of the basis.
- An a posteriori error estimator that certifies the RB solution with respect to the true approximation can be constructed.
- We have presented efficient methods for the basis construction.
- We have found complications on the neighborhoods of resonances.

# Conclusions and Future Work

## Future Work

- Use of different meshes for the primal and dual problems when computing the true approximation.
- Efficient and accurate way for computing $\tilde{\beta}^{pr}(\mu)$.
- Strategy for the resonances.
- Extension to nonaffine functionals and nonlinear problems.
- Possible extension to time-dependent wave propagation problems.
- Combination of reduced basis method with domain decomposition techniques (RB + Mortar, RB + DG).

# Appendix

# The Off-line Part of the Algorithm

Back

**The off-line computations: Done once and for all.**

- Construction of the reduced basis:

$$u(\mu_j) \equiv \xi_j, \quad j \in \{1, \ldots, N_{MAX}\}.$$

- Compute

$$f(\xi_i), \quad i \in \{1, \ldots, N_{MAX}\}.$$

- Compute the matrices $\forall\ q \in \{1, \ldots, Q_a\}$,

$$a_q(\xi_j, \xi_i), \quad (i, j) \in \{1, \ldots, N_{MAX}\}^2.$$

# The On-line Part of the Algorithm

Back

## The on-line computations: # of comp. independent on $\mathcal{N}$

- Ensembling the RB matrices:

$$\#Comp. \approx Q_a \times N^2.$$

- Ensembling the RB RHS and output terms:

$$\#\text{Comp.} \approx N.$$

- Solving the full linear systems for the primal and dual RB problems:

$$\#\text{Comp.} \approx N^3.$$

- Compute the RB output:

$$\#\text{Comp.} \approx N.$$

# Computation of $\varepsilon_N(\mu)$: Off-line part.

Back

**The off-line computations: Done once and for all.**

- Obtaining the following Riesz representation elements

$$\rho^X_{f(\cdot)},$$
$$\rho^X_{a_q(\xi_k,\cdot)}, \quad \forall\, (q,k) \in \{1,\ldots,Q_a\} \times \{1,\ldots,N_{MAX}\}.$$

- Compute the X-dot product of those elements:

$$(\rho^X_{f(\cdot)}, \rho^X_{f(\cdot)})_X,$$
$$(\rho^X_{a_q(\xi_k,\cdot)}, \rho^X_{a_{\tilde{q}}(\xi_{\tilde{k}},\cdot)})_X, \quad \forall\, (q,\tilde{q},k,\tilde{k}) \in \{1,\ldots,Q_a\}^2 \times \{1,\ldots,N_{MAX}\}^2,$$
$$(\rho^X_{f(\cdot)}, \rho^X_{a_{\tilde{q}}(\xi_k,\cdot)})_X, \quad \forall\, (\tilde{q},k) \in \{1,\ldots,Q_a\} \times \{1,\ldots,N_{MAX}\},$$

# Computation of $\varepsilon_N(\mu)$: On-line part.

Back

**The on-line computations: # of comp. independent on $\mathcal{N}$**

- The first term (involving the terms $f(\cdot)$):

$$\#Comp. \;\approx\; 1.$$

- The second term (involving the terms $a_q(\xi_k^{pr}, \cdot)$):

$$\#Comp. \;\approx\; Q_a^2 \times N^2.$$

- The third term (involving the crossed terms):

$$\#Comp. \;\approx\; Q_a \times N.$$

# Appendix

# Proof of the Well-Posedness of the RB Problem

Back

**Proof**

The existence and uniqueness are guaranteed by the discrete inf-sup parameter assumption. That condition also implies the existence of $\rho^{X_N}_{a(u_N(\mu),\cdot;\mu)} \in X_N$ /

$$\begin{aligned}\tilde{\beta}_0 \, \|u_N(\mu)\|_X \, \|\rho^{X_N}_{a(u_N(\mu),\cdot;\mu)}\|_X & \\ &\leq \left|a(u_N(\mu), \rho^{X_N}_{a(u_N(\mu),\cdot;\mu)}; \mu)\right| \\ &= \left|f(\rho^{X_N}_{a(u_N(\mu),\cdot;\mu)})\right| \\ &\leq \|f(\cdot)\|_{X'} \, \|\rho^{X_N}_{a(u_N(\mu),\cdot;\mu)}\|_X,\end{aligned}$$

which implies the continuity with respect to the RHS.

# Proof of the a Priori Estimates



**Proof**

Using the discrete inf-sup condition we know that for all $w \in X_N$ there exists $\rho^{X_N}_{a(u_N(\mu)-w,\cdot;\mu)} \in X_N$ such that

$$\begin{aligned} \tilde{\beta}_0(\mu)\ \|u_N(\mu) - w\|_X\ \|\rho^{X_N}_{a(u_N(\mu)-w,\cdot;\mu)}\|_X \ &\leq \\ |a(u_N(\mu) - w, \rho^{X_N}_{a(u_N(\mu)-w,\cdot;\mu)}; \mu)| \ &\leq \end{aligned}$$

# Proof of the a Priori Estimates

Back

**Proof**

Using the discrete inf-sup condition we know that for all $w \in X_N$ there exists $\rho^{X_N}_{a(u_N(\mu)-w,\cdot;\mu)} \in X_N$ such that

$$\begin{aligned} \tilde{\beta}_0(\mu)\ \|u_N(\mu) - w\|_X\ \|\rho^{X_N}_{a(u_N(\mu)-w,\cdot;\mu)}\|_X \ &\leq \\ |a(u\ \ (\mu) - w, \rho^{X_N}_{a(u_N(\mu)-w,\cdot;\mu)}; \mu)| \ &\leq \\ \gamma(\mu)\|u(\mu) - w\|_X\|\rho^{X_N}_{a(u_N(\mu)-w,\cdot;\mu)}\|_X, & \end{aligned}$$

where we've used the Galerkin orthogonality of $u_N(\mu) - u(\mu)$ on $X_N$.

# Proof of the a Priori Estimates



**Proof**

Simple computations show that

$$
\begin{aligned}
|s(\mu) - s_N(\mu)| &= |f(u(\mu) - u_N(\mu); \mu)| \\
&= |a(u_N(\mu) - u(\mu), u_N(\mu) - u(\mu); \mu)| \\
&\leq \gamma(\mu)\|u(\mu) - u_N(\mu)\|_X^2.
\end{aligned}
$$

We finally use the estimate on the reduced basis approximation to conclude.

Back

**Proof**

We have that

$$\begin{aligned}
\tilde{\beta}(\mu) &\leq \inf_{v\in X}\sup_{w\in X}\frac{|a(v,w;\mu)|}{\|v\|_X\|w\|_X} && (\ =\ \beta(\mu)\ ) \\
&\leq \sup_{w\in X}\frac{|a(u(\mu)-u_N(\mu),w;\mu)|}{\|u(\mu)-u_N(\mu)\|_X\|w\|_X} && (\ =\ \eta_N(\mu)\ \tilde{\beta}(\mu)) \\
&\leq \sup_{v\in X}\sup_{w\in X}\frac{|a(v,w;\mu)|}{\|v\|_X\|w\|_X} && (\ =\ \gamma(\mu)).
\end{aligned}$$

Dividing these inequalities by $\tilde{\beta}(\mu)$ we conclude the proof.

 Back

## Proof

We have that

$$\begin{aligned} |s_N(\mu) - s(\mu)| &= |a(u_N(\mu) - u(\mu), u_N(\mu) - u(\mu); \mu)| \\ &= |R(u_N(\mu) - u(\mu); \mu)| \\ &\leq \varepsilon_N(\mu)\ \|u_N(\mu) - u(\mu)\|_X. \end{aligned}$$

Using the inf-sup condition, $\exists\ \rho^{X_N}_{a(u_N(\mu)-u_(\mu),\cdot;\mu)} \in X$ such that

$$\begin{aligned} &\beta(\mu)\|u_N(\mu) - u(\mu)\|_X \|\rho^{X_N}_{a(u_N(\mu)-u_(\mu),\cdot;\mu)}\|_X \\ &\qquad \leq |a(u_N(\mu) - u(\mu), \rho^{X_N}_{a(u_N(\mu)-u_(\mu),\cdot;\mu)}; \mu)| \end{aligned}$$

Back

**Proof**

We have that

$$\begin{aligned} |s_N(\mu) - s(\mu)| &= |a(u_N(\mu) - u(\mu), u_N(\mu) - u(\mu); \mu)| \\ &= |R(u_N(\mu) - u(\mu); \mu)| \\ &\leq \varepsilon_N(\mu) \, \|u_N(\mu) - u(\mu)\|_X. \end{aligned}$$

Using the inf-sup condition, $\exists \ \rho^{X_N}_{a(u_N(\mu)-u_{(}\mu),\cdot;\mu)} \in X$ such that

$$\begin{aligned} \beta(\mu)\|u_N(\mu) - u(\mu)\|_X \|\rho^{X_N}_{a(u_N(\mu)-u_{(}\mu),\cdot;\mu)}\|_X \\ &\leq |a(u_N(\mu) - u(\mu), \rho^{X_N}_{a(u_N(\mu)-u_{(}\mu),\cdot;\mu)}; \mu)| \\ &= |R(\rho^{X_N}_{a(u_N(\mu)-u_{(}\mu),\cdot;\mu)}; \mu)| \\ &\leq \varepsilon_N(\mu) \, \|\rho^{X_N}_{a(u_N(\mu)-u_{(}\mu),\cdot;\mu)}\|_X. \end{aligned}$$

# Proof of the a Posteriori Error Estimator Inequalities

Back

## Proof

We have that

$$\begin{aligned} |s_N(\mu) - s(\mu)| &= |a(u_N(\mu) - u(\mu), u_N(\mu) - u(\mu); \mu)| \\ &= |R(u_N(\mu) - u(\mu); \mu)| \\ &\leq \varepsilon_N(\mu)\, \|u_N(\mu) - u(\mu)\|_X. \end{aligned}$$

We have thus

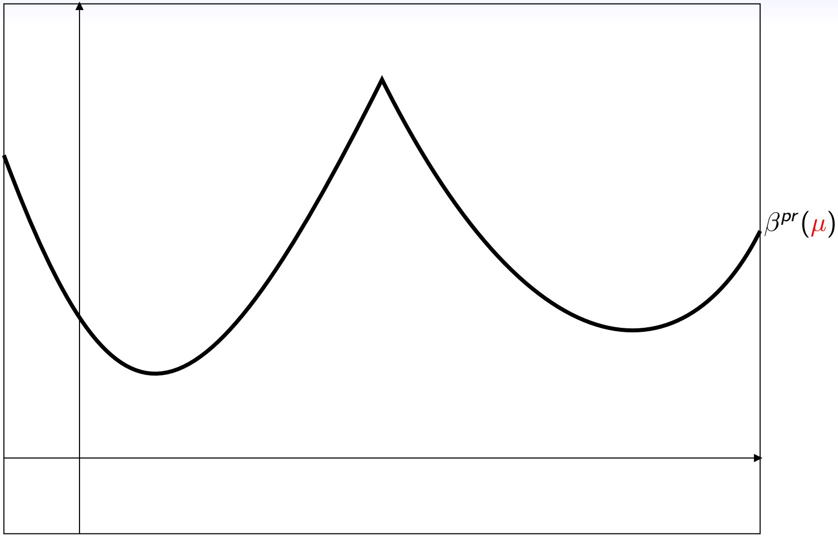
$$\|u_N(\mu) - u(\mu)\|_X \leq \frac{\varepsilon_N(\mu)}{\tilde{\beta}(\mu)}.$$

We combine both inequalities to conclude the proof.

# Appendix

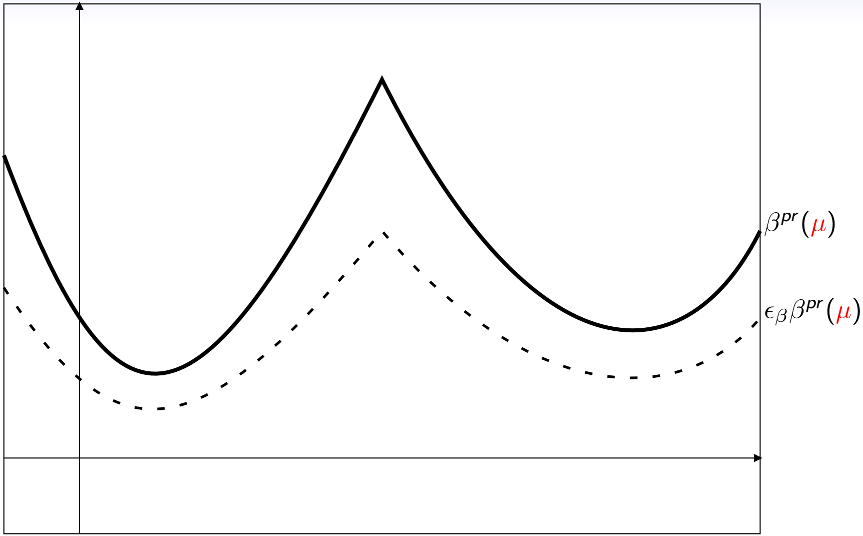
# A Posteriori Theory: How to Compute $\tilde{\beta}^{pr}(\mu)$





# A Posteriori Theory: How to Compute $\tilde{\beta}^{pr}(\mu)$

Back



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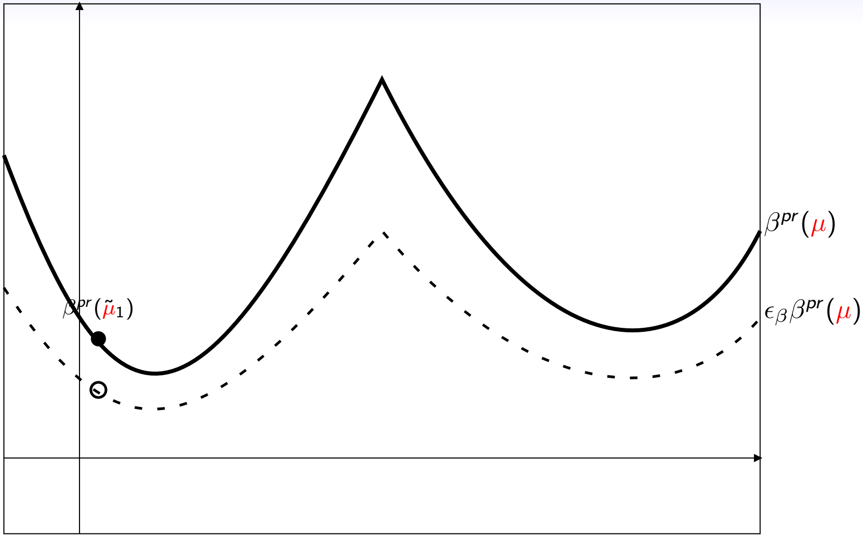

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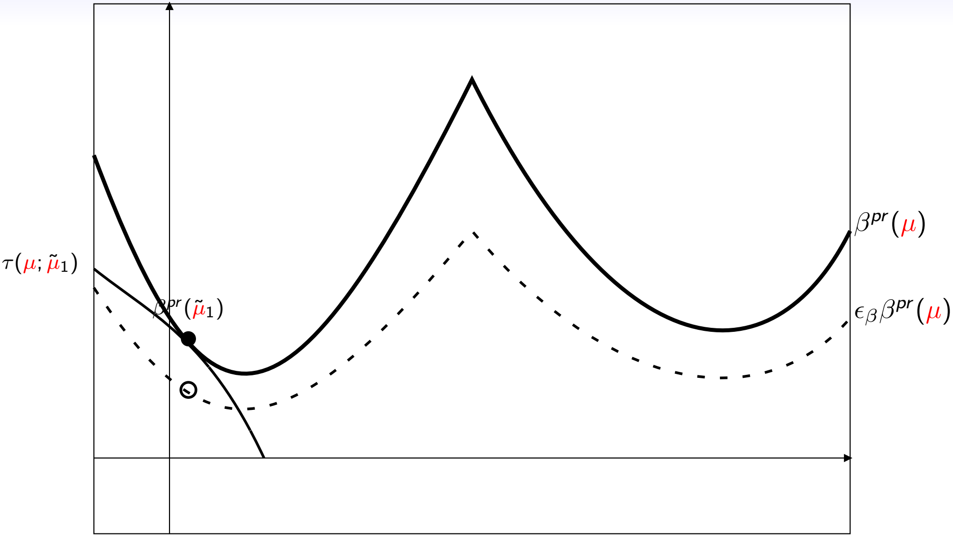

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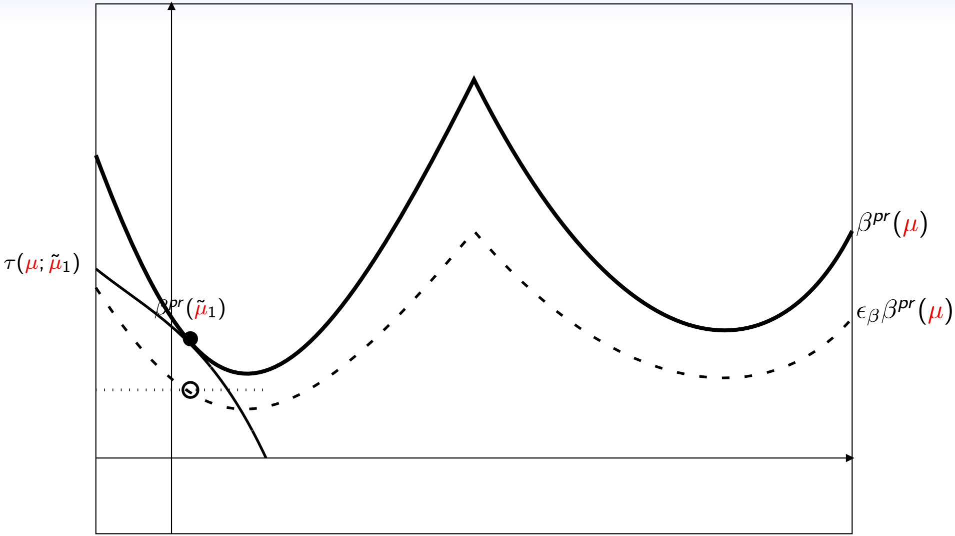

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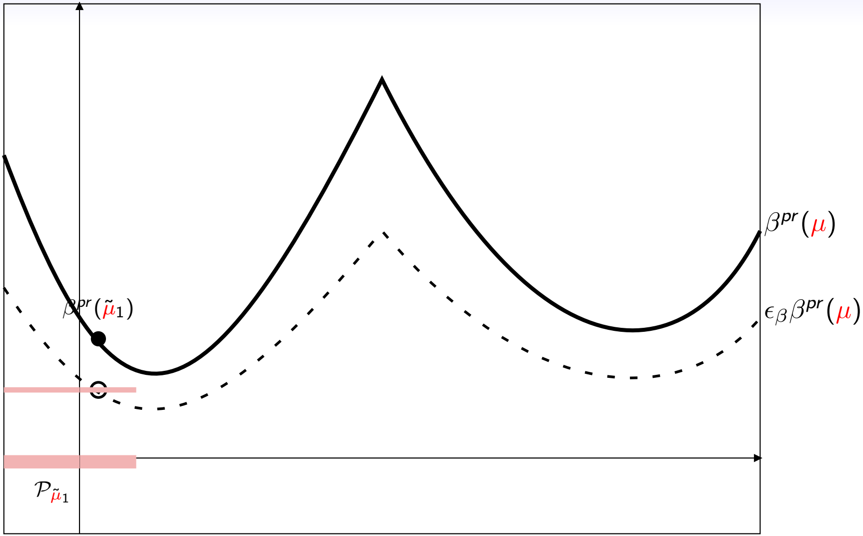

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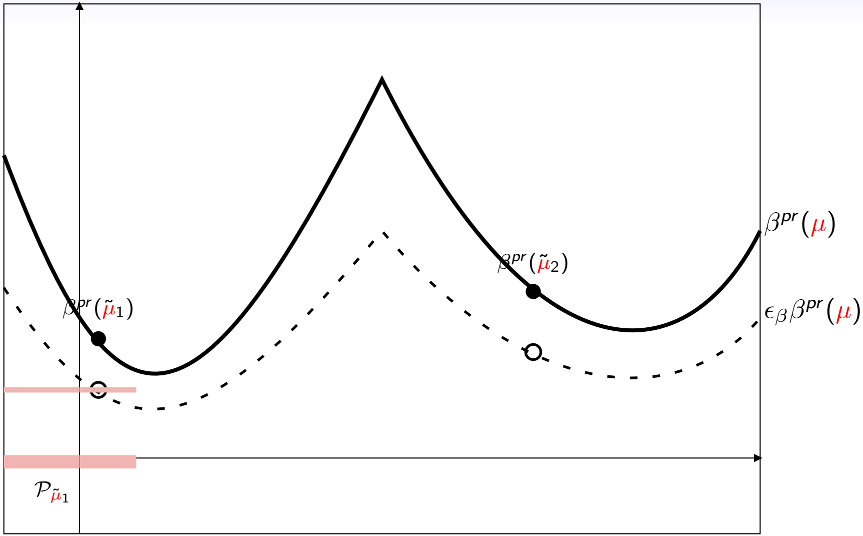

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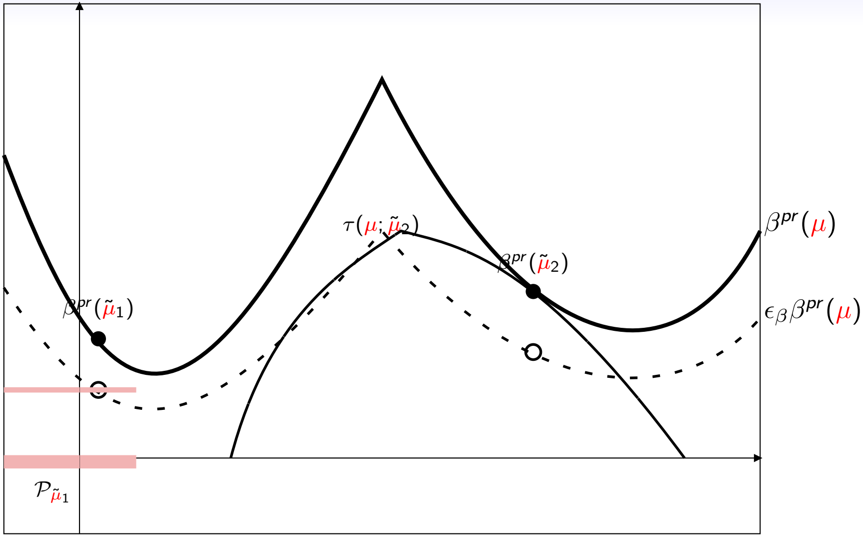

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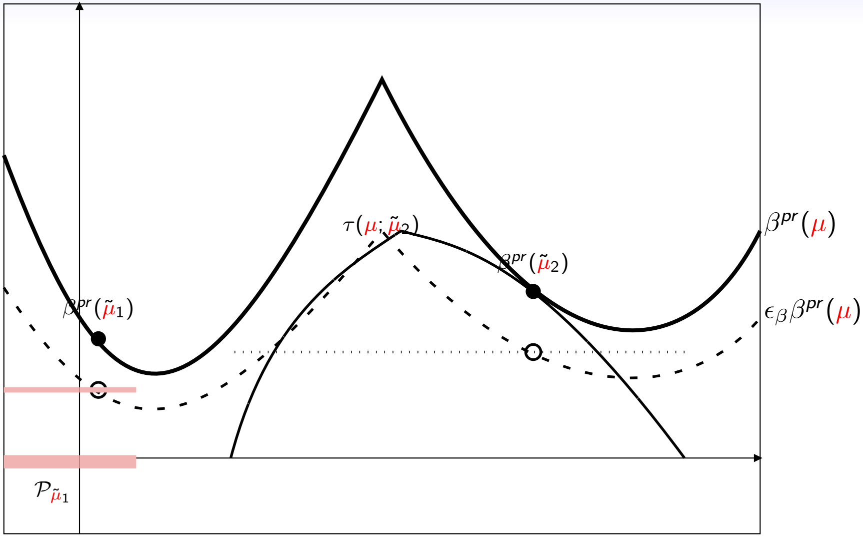

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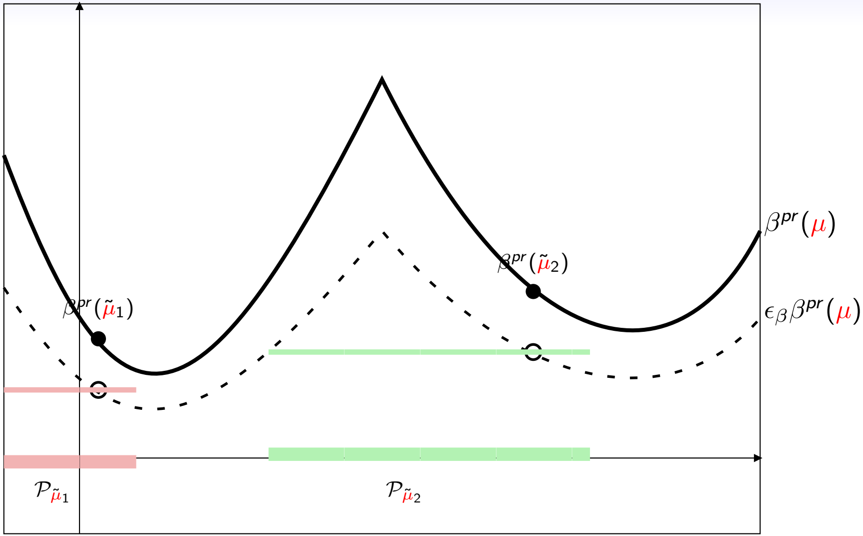

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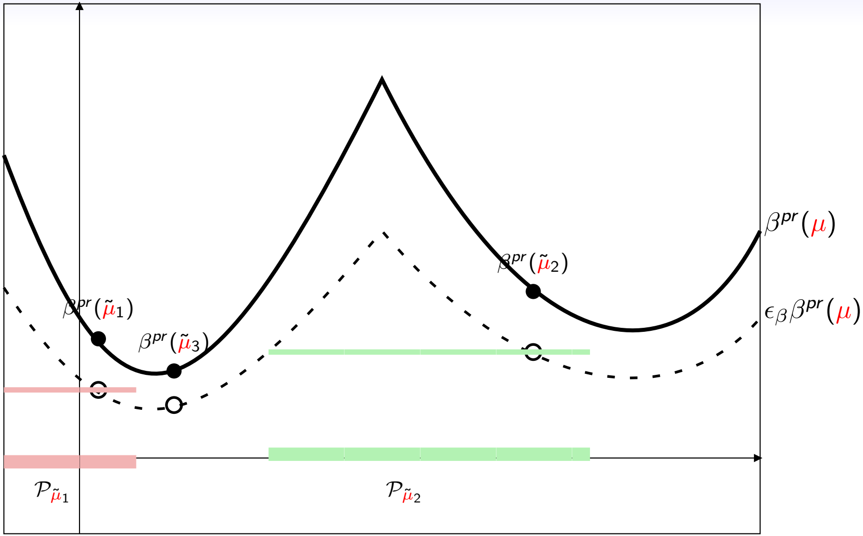

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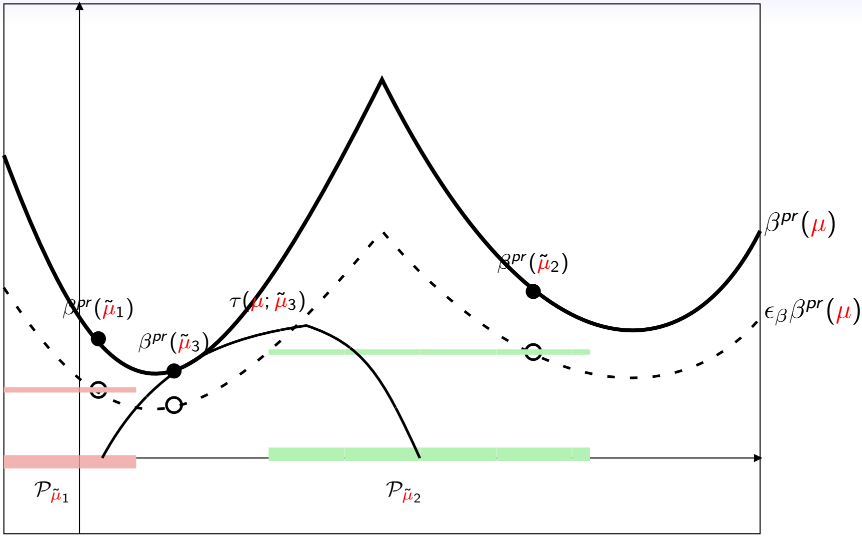

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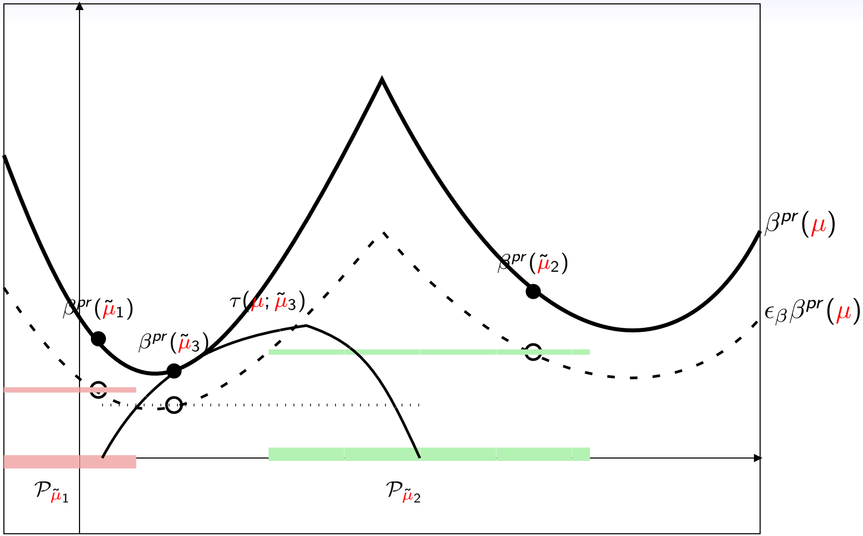

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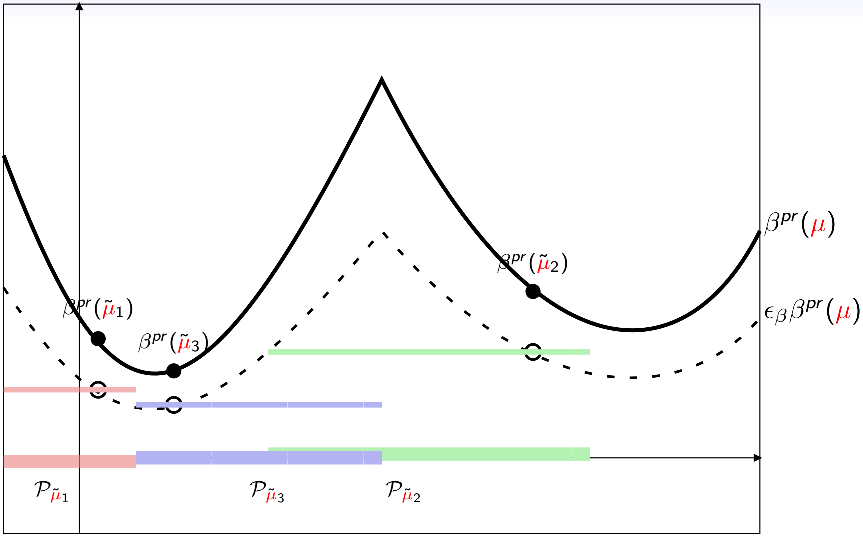

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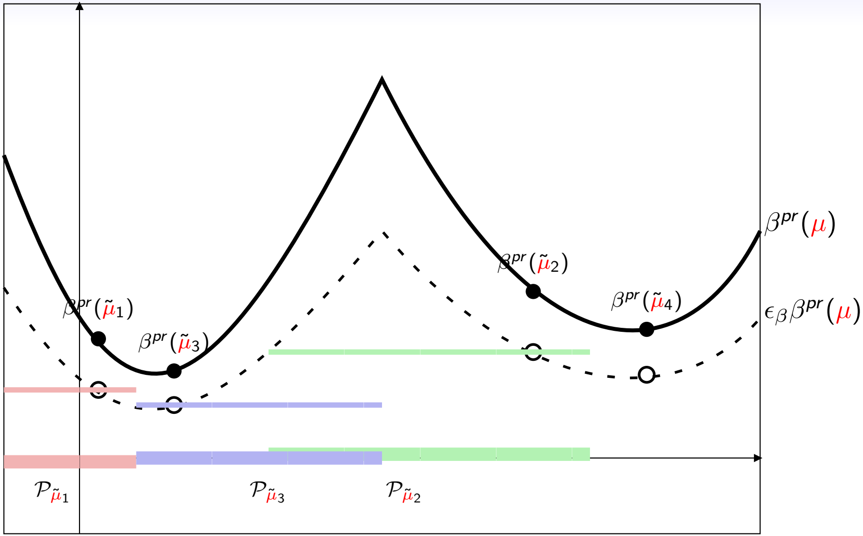

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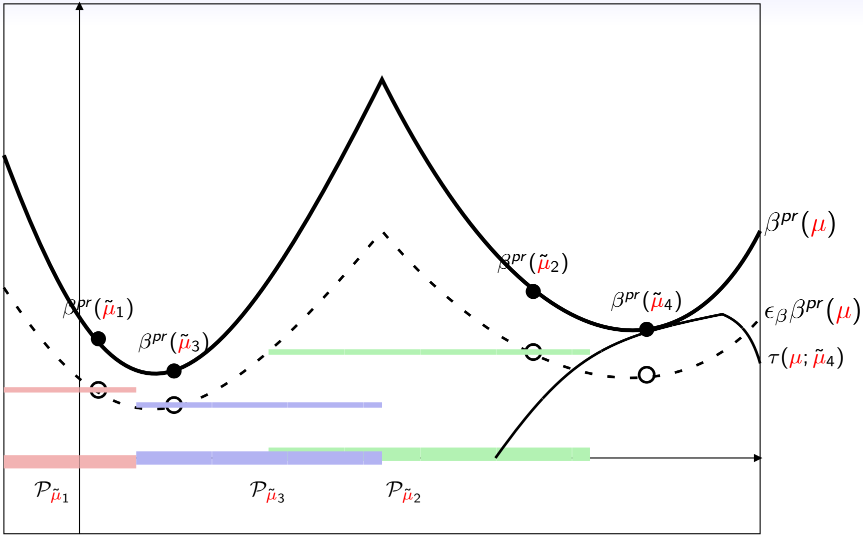

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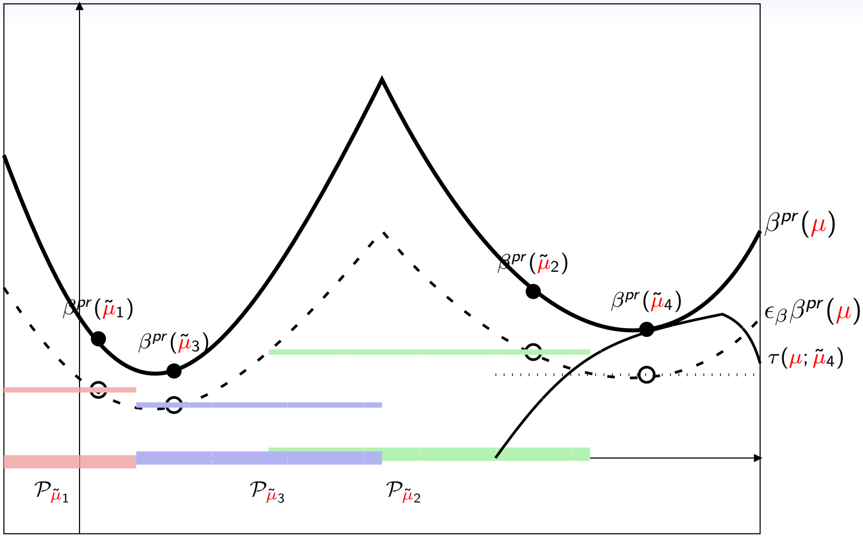

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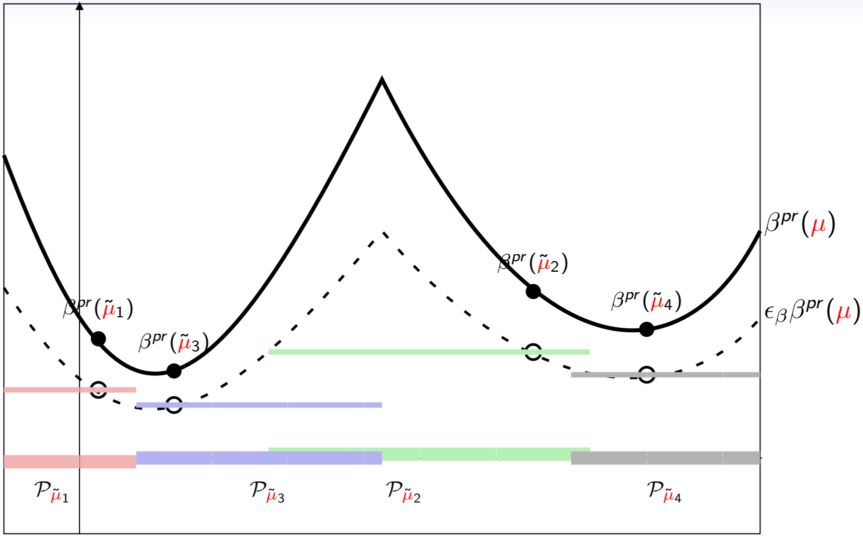

# Appendix

# The Riesz Representation Theorem

Back to $\varepsilon_N(\mu)$ Back to $\tilde{\beta}(\mu)$

## Theorem (Riesz)

*Let $f(\cdot)$ be a continuous linear functional from a $K$-Hilbert space $H$ (into the field $K$). Then, there exists a unique $\rho^H_{f(\cdot)} \in H$ such that*

$$f(x) = (\rho^H_f, x)_H, \quad \forall\, x \in H,$$

*where $(\cdot,\cdot)_H$ denotes the inner product on $H$. Moreover, we have that*

$$\|f(\cdot)\|_{H'} := \sup_{x\in H} \frac{|f(x)|}{\|x\|_X} = \|\rho^H_{f(\cdot)}\|_X,$$

$$\rho^H_f(\cdot) = \arg\sup_{x\in H} \frac{|f(x)|}{\|x\|_X}.$$

# Appendix

# The True Approximation: Discontinuous Galerkin

Back

- We rewrite Maxwell equations as a first order system introducing the magnetic field $H_z = \frac{1}{i\mu\omega}\text{curl } E$:

$$\left| \begin{array}{rcrcr} i\varepsilon\omega E_x & = & \frac{\partial H_z}{\partial y} & - & J_x, \\ i\varepsilon\omega E_y & = & -\frac{\partial H_z}{\partial x} & - & J_y, \\ i\mu\omega H_z & = & \frac{\partial E_x}{\partial y} & - & \frac{\partial E_y}{\partial x}. \end{array} \right.$$

# The True Approximation: Discontinuous Galerkin

Back

- We introduce a mesh of the computational domain:

$$\Omega = \bigcup_k D^k.$$

- We introduce the approximation space

$$X = \{\phi \equiv (\phi_1, \phi_2, \phi_3) \in (L^2(\Omega))^3 \text{ such that } \phi_{|D^k} \in (\mathcal{P}_p(D^k))^3\},$$

- We consider the approximate solution

$$(E_x, E_y, H_z) \in X.$$

# The True Approximation: Discontinuous Galerkin

Back

- We multiply the equations by a test function $\phi \in X$.
- We integrate on an element $D^k$.

$$
\left|
\begin{aligned}
(i\varepsilon\omega E_x, \phi_1)_{D_k} &= (\frac{\partial H_z}{\partial y}, \phi_1)_{D_k} - (J_x, \phi_1)_{\delta D_k}, \\
(i\varepsilon\omega E_y, \phi_2)_{D_k} &= -(\frac{\partial H_z}{\partial x}, \phi_2)_{D_k} - (J_y, \phi_2)_{\delta D_k}, \\
(i\mu\omega H_z, \phi_3)_{D_k} &= (\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}, \phi_3)_{D_k}.
\end{aligned}
\right.
$$

# The True Approximation: Discontinuous Galerkin

Back

- We multiply the equations by a test function $\phi \in X$.
- We integrate on an element $D^k$.
- We apply the Green's formula. At the interfaces we have two values. We have to define those traces.

$$
\begin{aligned}
(i\varepsilon\omega E_x, \phi_1)_{D_k} &= -(H_z, \frac{\partial \phi_1}{\partial y})_{D_k} + (\widehat{H}_z\ n_y, \phi_1)_{\delta D_k} \\
&\quad - (J_x, \phi_1)_{\delta D_k}, \\
(i\varepsilon\omega E_y, \phi_2)_{D_k} &= (H_z, \frac{\partial \phi_2}{\partial x})_{D_k} - (\widehat{H}_z\ n_x, \phi_2)_{\delta D_k} \\
&\quad - (J_y, \phi_2)_{\delta D_k}, \\
(i\mu\omega H_z, \phi_3)_{D_k} &= -(E_x, \frac{\partial \phi_3}{\partial y})_{D_k} + (\widehat{E}_x\ n_y, \phi_3)_{\delta D_k} \\
&\quad +(E_y, \frac{\partial \phi_3}{\partial x})_{D_k} - (\widehat{E}_y\ n_x, \phi_3)_{\delta D_k}.
\end{aligned}
$$

# The True Approximation: Discontinuous Galerkin

## The traces: Central fluxes + stabilization terms

We use central fluxes with a stabilization term:

$$\left| \begin{array}{lll} \widehat{E}_x & := & \dfrac{E_x^+ + E_y^-}{2}, \qquad \widehat{E}_y := \dfrac{E_y^+ + E_y^-}{2}, \\ \widehat{H}_z & := & \dfrac{H_z^+ + H_z^-}{2} + \tau \left( E_x^+ \, n_y^+ + E_x^- \, n_y^- - \right. \\ & & \left. E_y^+ \, n_x^+ - E_y^- \, n_x^- \right) \end{array} \right.$$

The stabilization parameter $\tau$ is given by:

$$\tau = \alpha \frac{p(p+1)}{\text{meas } e}.$$