# Recent progress on two vector bundles

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If you cover a space by open sets, a continuous function can be given by defining the function on each open set and demanding that the values agree at intersection points. A vector bundle can be defined by defining transition isomorphisms on intersections and demanding that they agree on triple intersections. A two-vector bundle is a natural extrapolation of these ideas, where the isomorphisms occur at triple intersections and the cocycle condition on quadruple intersections. In fact, two-vector bundles is what you get if you - when defining vector bundles - replace the ring of complex numbers by the category of finite dimensional complex vector spaces with sum and tensor as operations. Otherwise said, they are to vector bundles what gerbes are to line bundles.

The strange fact is that this very naïve setup has homotopy theoretical content. The chromatic picture gives a hierarchy of cohomology theories according to how deep structure of stable homotopy theory the cohomology theory detects. Chromatic filtration zero, one and infinity have nice geometric interpretations through functions, vector bundles and bordisms, and the geometric origin is important for the analysis of key problems. Such geometric interpretations have been missing in higher finite filtrations.

Elliptic cohomology and topological modular are examples at chromatic filtration two, and Segal conjectured that there ought to be a geometric interpretation of elliptic cohomology through quantum field theories.

In a recent paper, Baas, D, Richter and Rognes prove that two-vector bundles give rise to a cohomology theory which is represented by the algebraic K-theory of topological K-theory $ku$, which by results of Ausoni is of (a connected version of) chromatic filtration two. Hence we have a naturally defined geometric theory of the desired sort.

The connection to quantum field theories and the set-up of Stolz and Teichner is, however, still mysterious. There was a hope that an "integration of determinants through loops" construction would give a functor from two-vector bundles to quantum field theories, but this is unfortunately not the case.

Whereas commutative rings support determinants, this is not (in the most naïve sense) true for commutative ring spectra. In fact, neither the sphere spectrum nor topological K-theory supports determinants. The latter is important for us since it rules out the conjectured functor to quantum field theories.

However the reason for its failure is very interesting: it stems from an observation of Rognes that Ausoni's calculations of $K(ku)$ implies that the group of gerbes on the three dimensional sphere do not split off as a direct summand of the group of "virtual" two-vector bundles. This leaves one speculating about the geometry of two-vector bundles, even over very simple spaces.

Rationally, this problem vanishes, and Ausoni and Rognes has pushed through the program in this case: giving a virtual two-vector bundle on $X$ is rationally the same as giving its virtual "dimesion bundle" and an "anomality bundle" on the free loop space of $X$.

Two-vector bundles is a theory in its infancy, and much is still left to explore. In particular the geometric and analytic aspects are so far largely terra incognita.

In my talk I will try to explain some of these ideas and results.