Lecture on Gersten's conjecture

Satoshi Mochizuki
Toyo Universitiy, Japan

I intend to give a lecture on Gersten's conjecture. We recognize Gersten's conjecture as comparison problem between topological and γ-filtrations. In the authors view point, the difficulty of the problem is completely decomposing into two aspects. The first aspect is homotopy theoretic difficulty. The second one is commutative ring theoretic. I will give a strategy how to overcome these difficulty and partially affirmative answers. More precisely, I have the following plan.

1. For the first aspect, I think the problem is lack of fundamental techniques on algebraic K-theory. More precisely, lack of the following techniques:

   1.1. Comparison techniques of two K-theory of Waldhausen categories.

   1.2. Comparison techniques of maps between two K-theories.

   1.3. To attack the problem 1.1., I examined a mapping cone. More precisely, for connected K-theory, we can describe the mapping cone of the map associated to an exact functor between Waldhausen categories. I proved the additivity theorem for f-construction in [Dev] which enables us to simplify the description of the mapping cone. By the virtue of this result, I got a new sufficient conditions that vanishing of the cone in [Dev] and [Kos]. I will give a survey of these results.

      1.3.1. Combining the technique 1.3. with the theory about semi-direct products of exact categories established in [PID] and [Kos], we get a comparison theorem and a fibration theorem without assuming the factorization axiom. I will explain the results about these topics.

   1.4. Relating results with 1.2. is the following:

1.4.1. The universal property of algebraic K-theory associated to semi-simple exact categories established by Quillen-Sherman.

1.4.2. Kliesli-Thomason rectification theory for exotic functors.

1.4.3. I recognized the construction of Sherman's universal map in 1.4.1. as a consequence of more general result about behaviour of internal Hom through the Cartesian closed functors and found a possibility of combining the techniques 1.4.1. with 1.4.2. by introducing notions of lax group representations and lax algebraic K-theory. In fact, improving the proof of Waldhausen's $s = iS$ theorem by utilizing the rectification theory, I proved the retraction theorem which says that usual algebraic K-theory is a retraction of lax algebraic K-theory. As its consequence, we can use lax group representations to 1.4.1. and it is a very applicative technique. I will give a survey about these topics.

2. For the second aspect, I think the problem is lack of the structure theorem about coherent Modules over a general regular local ring like as an elementary divisor theory or Bourbaki- Iwasawa-Serre structure theorem of coherent Modules over two dimensional regular local rings.

   2.1. To establish the structure theorem above, we define a notion of pure weight perfect Modules on schemes in [DMTI]. I will explain the notion and fundamental examples. An important result is that for divisorial schemes, up to derived Morita equivalence, the DG-category of perfect complexes supported on a regular closed immersion is generated by certain pure weight perfect Modules.

   2.2. To build up the structure theorem, we will first replace pure weight Modules with appropriate Koszul cubes defined in [Kos]. Important result is that the K-theory of certain exact category of pure weight perfect Modules is homotopy equivalent to the K-theory of appropriate Waldhausen category of Koszul cubes. I will give a lecture about these topics.

3. I will state the local Gersten's conjecture which implies original Gersten's conjecture and explain an idea about how to prove the local Gersten's conjecture by utilizing the results above.

   3.1. In particular, I intend to prove the local Gersten's conjecture for regular system of parameters which is a generalization of the

result in [DVR]. Moreover if my progressing work is completely finish, I will be able to give a more strong result.

References

[DVR] S. Mochizuki, *Gersten's conjecture for commutative discrete valuation rings*, available at `http://www.math.uiuc.edu/K-theory/0819/`.

[PID] S. Mochizuki, *Higher algebraic K-theory of finitely generated torsion modules over principal ideal domains*, available at `http://www.math.uiuc.edu/K-theory/0823/`.

[Dev] S. Mochizuki, *A dévissage theorem for modular exact categories with weak equivalences*, available at `http://www.math.uiuc.edu/K-theory/0882/`.

[DMTI] T. Hiranouchi and S. Mochizuki, *Deforming motivic theories I: Pure weight perfect modules on divisorial schemes*, available at `http://www.math.uiuc.edu/K-theory/0885/`.

[Kos] S. Mochizuki, *Higher K-theory of Koszul cubes*, available at `http://www.math.uiuc.edu/K-theory/0888/`.