Distortions and bidistortions

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The idea of "distortion" of a morphism between objects of an "higher level" category is formalized in a theory.

Introduction

This talk will be about a theory produced after several abstractions, beginning with the study of the geometry of leaves of foliated spaces [1]. It has the flavor of a combination of categories and metrics. There may be already combinations of this kind, but we hope that our point of view is new. The study of foliated spaces seems to be only one of the many applications that this very general theory could have.

1 Distortions

An *underlying functor* (or *u-functor*) $\mathcal{A}/\mathcal{B}$ is a functor $|\ |:\mathcal{A}\to\mathcal{B}$ whose restriction $\mathcal{A}(M,N)\to\mathcal{B}(|M|,|N|)$ is injective $\forall M,N\in\mathcal{A}_0$. Thus we can consider $\mathcal{A}$ as a subcategory of the category $\widetilde{\mathcal{B}}$ with $\widetilde{\mathcal{B}}_0=\mathcal{A}$, $\widetilde{\mathcal{B}}(M,N)\equiv\mathcal{B}(|M|,|N|)$ and the operation induced by $\mathcal{B}$. The typical example of u-functor is provided by two species of structures, one subordinated to the other, with corresponding classes of morphisms.

Consider the following pre-order relation between sets of functions $X\to[0,\infty]$: $C\preccurlyeq D$ if, $\forall\epsilon>0$, $\exists\delta>0$ such that,

$$d(x)<\delta\ \forall d\in D,\ \forall x\in X\implies c(x)<\epsilon\ \forall c\in C,\ \forall x\in X\ .$$

A *distortion* of a u-functor $\mathcal{A}/\mathcal{B}$ is a map $\mathfrak{d}:\widetilde{\mathcal{B}}_1\to[0,\infty]$ satisfying $\mathfrak{d}^{-1}(0)=\mathcal{A}_1$ and $\mathfrak{d}(f_n\cdots f_1)\preccurlyeq\mathfrak{d}(f_1)+\cdots+\mathfrak{d}(f_n)$. For $M,N\in\mathcal{A}_0$, let $\mathfrak{d}_{M,N}=\mathfrak{d}|_{\widetilde{\mathcal{B}}(M,N)}$. An induced distortion $\hat{\mathfrak{d}}$ is defined by $\hat{\mathfrak{d}}(f)=\inf\{\mathfrak{d}(f_1)+$

$\cdots + \mathfrak{d}(f_n) \mid f_n \cdots f_1 = f\}$. Then a pseudo-metric $d_{\mathfrak{d}}$ on $\mathcal{A}_0$ (with possible infinite values) is given by $d_{\mathfrak{b}}(M,N) = \|(\inf \mathfrak{d}_{M,N}, \inf \mathfrak{d}_{N,M})\|$. It induces a pseudo-metric on the quotient set $\mathcal{A}\backslash\mathcal{A}_0$ of $\mathcal{A}_0$ defined by the equivalence relation $M \sim N$ when $\mathcal{A}(M,N) \neq \emptyset \neq \mathcal{A}(N,M)$. For instance, the best Lipschitz constant defines a distortion of Met / Set.

2 Bidistortions

An *involutive category* is a category $\mathcal{C}$ endowed with an *involution* (a contravariant functor $I : \mathcal{C} \to \mathcal{C}$ such that $I_0 = \mathrm{id}_{\mathcal{C}_0}$ and $I^2 = \mathrm{id}_{\mathcal{C}}$). The notation $\bar{f} = I(f)$ and $\overline{S} = \{\bar{f} \mid f \in S\}$ will be used. For example, any groupoid is involutive with $\bar{f} = f^{-1}$. Also, any category $\mathcal{B}$ induces an involutive category $\mathcal{B}^{\mathrm{in}}$ with $\mathcal{B}^{\mathrm{in}}_0 = \mathcal{A}_0$, $\mathcal{B}^{\mathrm{in}}(M,N) \equiv \mathcal{B}(M,N) \times \mathcal{B}(N,M)$, the operation induced by $\mathcal{B}$ and $\overline{(f,g)} = (g,f)$. The concept of *involutive* functor is now straightforward. Any u-functor $\mathcal{A}/\mathcal{B}$ induces an involutive u-functor $\mathrm{Iso}(\mathcal{A})/\mathcal{B}^{\mathrm{in}}$ given by $\phi \mapsto (|\phi|, |\phi^{-1}|)$.

A *crossing* of a category $\mathcal{D}$ is mapping $S \mapsto S^\times$, for sets S of parallel morphisms of $\mathcal{D}$, so that $\emptyset^\times = \emptyset$, $S \subset S^\times$ and $S \subset T \Rightarrow S^\times \subset T^\times$. For instance, $\mathcal{B}^{\mathrm{in}}$ has a crossing given by

$$S^\times = \{(f,g') \mid (f,g),(f',g') \in S \text{ for some } f',g \in \mathcal{B}_1\} \;.$$

Let $\mathcal{C}/\mathcal{D}$ be an involutive u-functor, and suppose that $\mathcal{D}$ is endowed with a crossing. A *bidistortion* $\mathfrak{b}$ of $\mathcal{C}/\mathcal{D}$ is a distortion satisfying:

$$\sup \mathfrak{b}(\{\sigma_1,\sigma_n\}^\times) \preccurlyeq \sup \mathfrak{b}(\{\sigma_1,\sigma_2\}^\times) \cup \{\sigma_2,\sigma_3\}^\times \cup \cdots \cup \{\sigma_{n-1},\sigma_n\}^\times) \;,$$
$$\sup \mathfrak{b}(\{\bar{\sigma}\sigma, \mathrm{id}\}^\times) \preccurlyeq \mathfrak{b}(\sigma) \;, \qquad \sup \mathfrak{b}(\overline{S}^\times) \preccurlyeq \sup \mathfrak{b}(S^\times) \;, \qquad \sup \mathfrak{b}(\{\sigma\}^\times) \preccurlyeq \mathfrak{b}(\sigma) \;,$$
$$\sup \mathfrak{b}(S^\times) \preccurlyeq \sup \bigcup_{\sigma,\tau \in S} \mathfrak{b}(\{\sigma,\tau\}^\times) \;, \qquad \sup \mathfrak{b}((TS)^\times) \preccurlyeq \sup \mathfrak{b}(T^\times S^\times) \;.$$

If $\sup \mathfrak{b}(S)$ is small, consider $\sup \mathfrak{b}(S^\times)$ as some kind of "diameter" of S. With this idea, we introduce the concepts of (*a-commutative*) *a-functor* and *a-natural transformation*, where "*a*-" stands for *asymptotically* or *asymptotic*. They are defined as maps that "become" (commutative) functors or natural transformations at certain limit. In the same way, we introduced the equivalence relation of being *asymptotic* between a-natural transformations, whose equivalence classes form a groupoid $\overline{\mathcal{C}}$, called the *completion* of $\mathcal{C}$, and there is a canonical functor $\iota : \mathcal{C} \to \overline{\mathcal{C}}$. It is said that $\mathfrak{b}$ is *complete* if $\bar{\iota} : \mathcal{C}\backslash\mathcal{C}_0 \to \overline{\mathcal{C}}\backslash\overline{\mathcal{C}}_0$ is bijective.

Endow $\mathcal{C}_0$ with $d_{\mathfrak{b}}$, and $\overline{\mathcal{C}}_0$ with another pseudo-metric $\bar{d}_{\mathfrak{b}}$, similarly defined, which induces a pseudo-metric on $\overline{\mathcal{C}}\backslash\overline{\mathcal{C}}_0$. It turns out that $\iota_0 : \mathcal{C}_0 \to \overline{\mathcal{C}}_0$ is an isometry with dense image and $\overline{\mathcal{C}}_0$ is complete. $\mathcal{C}\backslash\mathcal{C}_0$ and $\overline{\mathcal{C}}\backslash\overline{\mathcal{C}}_0$ are Hausdorff when some kind of compactness or pre-compactness is satisfied.

3 Examples and applications

Any metric can be considered as a bidistortion. Bidistortions can be used to produce interesting spaces, like the Gromov space of isometry classes of pointed proper metric spaces, its subspace defined by complete connected Riemannian manifolds, and many "universal" foliated spaces. We also hope that bidistortions can be used to generalize our results on the generic geometry of leaves [1].

References

[1] J.A. Álvarez López & A. Candel. *Generic Geometry of Leaves*. To appear.