Obtaining the Postnikov system of an arbitrary CW-complex algebraically

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We determine the Postnikov tower of an arbitrary space X (CW-complex) as the geometric realization of the Postnikov tower of $\Pi(X)$, the fundamental simplicial groupoid of the singular complex of X, which is obtained in the category of simplicial groupoids by a purely algebraic method similar to the one used in [2] for crossed complexes. Eventually we obtain the Postnikov invariants from the fibrations in the Postnikov tower of $\Pi(X)$ (which are shown to represent cocycles in an algebraic cohomology) by defining an appropriate connecting map from the algebraic cohomology to singular cohomology.

Introduction

The Postnikov tower of a space X is a diagram of spaces $\cdots \to X_{n+1} \xrightarrow{\eta_{n+1}} X_n \xrightarrow{\eta_n} X_{n-1} \to \cdots \to X_0$, such that: (1) its limit is a space which is weak equivalent to X, (2) for each $n \geq 1$, X_n is a topological n-type, and (3) for each $n \geq 1$, η_n is a fibration whose fibres are Eilenberg-MacLane space. Associated with each fibration η_n there is a cohomology element $k_n \in H^{n+1}_{\text{sing}}\big(X_{n-1}, \pi_n(X)\big)$ in the singular cohomology of CW-complexes which is called the n^{th} Postnikov invariant of X and informs us about how the fibres of η_n are glued together to form X_n in the sense that $X_n = X_{n-1} \times_{k_n} K\big(\pi_n(X), n\big)$.

In the paper, [2], previously presented in a short communication by E. Faro in Seca I, we introduced a general algebraic procedure to obtain Postnikov towers and Postnikov invariants and developed it in the category of crossed complexes providing an algebraic calculation of the Postnikov towers and Postnikov invariants of filtered spaces which have the homotopy type of a crossed complex. In order to eliminate this restriction on the homotopy type of the spaces, we shall develop our algebraic approach in the category of simplicial groupoids, a category which models all homotopy types of CW-complexes.

1 Postnikov tower

The general scheme for the algebraic calculation of Postnikov towers introduced in [2] consists in "translating" the problem from a category $\mathcal{T}$ of CW-complexes to a "more algebraic" category Gd, the category of simplicial groupoids, which is related to $\mathcal{T}$ by means of an adjoint pair of functors *"fundamental simplicial groupoid of the singular complex of a space"*, $\Pi : \mathcal{T} \to \mathsf{Gd}$, and *"geometric realization of (the nerve of) a simplicial groupoid"*, $B : \mathsf{Gd} \to \mathcal{T}$, which induce an equivalence in the corresponding homotopy categories. Then, the geometric realization of the Postnikov tower of $\Pi(X)$ will be the Postnikov tower of X.

The calculation of the Postnikow towers of simplicial groupoids will proceed as follows: For each $n \geq 0$ we define, a subcategory Gd_n of Gd so that trivially $\mathsf{Gd}_n \subset \mathsf{Gd}_{n+1}$, an endofunctor $P_n \colon \mathsf{Gd} \to \mathsf{Gd}$ such that for every $\boldsymbol{\mathcal{G}} \in \mathsf{Gd}$, $P_n(\boldsymbol{\mathcal{G}}) \in \mathsf{Gd}_n$, P_n leaves Gd_n fixed, and $P_n P_{n+1} = P_n$, and a natural transformation $\delta^{(n)} \colon 1_{\mathsf{Gd}} \to P_n$ such that $\delta^{(n)} * P_n = P_n * \delta^{(n)} = 1_{P_n}$. Then we prove that for every simplicial groupoid $\boldsymbol{\mathcal{G}}$, the map $\eta^{(n+1)}_{\boldsymbol{\mathcal{G}}} = \delta^{(n)}_{P_{n+1}(\boldsymbol{\mathcal{G}})}$ is a fibration whose fibres have the homotopy type of a $K(\Pi, n+1)$, for each $n \geq 0$, and the cone formed by the maps $\delta^{(n)}_{\boldsymbol{\mathcal{G}}}$ over the chain of the $\eta^{(n+1)}_{\boldsymbol{\mathcal{G}}} = \delta^{(n)}_{P_{n+1}(\boldsymbol{\mathcal{G}})}$ is a limit diagram.

2 Postnikov invariants

The situation with respect to the Postnikov invariants is a little more involved. In the Postnikov tower of a space we have, for each n a fibration, η_{n+1}, from a $(n+1)$-type to a n-type. With a slight abuse of language and taking Gd_n as our category of n-types we can say that η_{n+1} is a fibration from a groupoid in Gd_n to an object in Gd_n. This, very closely resembles the concept of a 2-torsor in Gd_n, a gadget used by Duskin to represent 2-dimensional cotriple cohomology. In fact, if one makes a slightly different description of Duskin's 2-torsors (as given, for example, in [2, Appendix]) it is easy to show that by regarding the objects in Gd_{n+1} as groupoids in Gd_n, the fibration η_{n+1} becomes exactly a 2-torsor in Gd_n. Thus, it seems that η_{n+1} should determine in a natural way an element of an algebraic 2-cohomology in Gd_n. It only remains to connect the resulting 2-cohomology with the $(n+2)$-cohomology in the category of all homotopy types, where the actual Postnikov invariant is supposed to live.

References

[1] M. Bullejos, E. Faro, and M. A. Garcia-Muñoz. Homotopy colimits and cohomology with local coefficients. *Cahiers Topologie Géom. Différentielle Catég.*, 44 (1):63–80, 2003.

[2] M. Bullejos, E. Faro, and M. A. Garcia-Muñoz. Postnikov invariants of crossed complexes. *Journal of Algebra*, 285:238–291, 2005.

[3] M. A. García-Muñoz. *Un acercamiento algebraico a la teoría de torres de Postnikov.* Ph.D. thesis, Universidad de Granada, 2003. ISBN: 9788433842206. Granada, 2007. http://adrastea.ugr.es/record=b1655506*spiG