Generalized down-up algebras and their automorphisms

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A generalization of down-up algebras was introduced by Cassidy and Shelton in [4], the so-called generalized down-up algebras. We describe the automorphism group of conformal Noetherian generalized down-up algebras $L(f, r, s, \gamma)$ such that r is not a root of unity, listing explicitly the elements of the group. Then we apply these results to Noetherian down-up algebras, thus obtaining a characterization of the automorphism group of Noetherian down-up algebras $A(\alpha, \beta, \gamma)$ for which the roots of the polynomial $X^2 - \alpha X - \beta$ are not both roots of unity.

Introduction

Generalized down-up algebras were introduced by Cassidy and Shelton in [4] as a generalization of the down-up algebras $A(\alpha, \beta, \gamma)$ of Benkart and Roby [2]. Generalized down-up algebras include all down-up algebras, the algebras *similar to the enveloping algebra of* $\mathfrak{sl}_2$ defined by Smith [10], Le Bruyn's *conformal* $\mathfrak{sl}_2$ *enveloping algebras* [8] and Rueda's algebras *similar to the enveloping algebra of* $\mathfrak{sl}_2$ [9].

Two of the most remarkable examples of down-up algebras are $U(\mathfrak{sl}_2)$ and $U(\mathfrak{h})$, the enveloping algebras of the 3-dimensional complex simple Lie algebra $\mathfrak{sl}_2$ and of the 3-dimensional nilpotent, non-abelian Heisenberg Lie algebra $\mathfrak{h}$, respectively. These algebras have a very rich structure and representation theory which has been extensively studied, having an unquestionable impact on the theory of semisimple and nilpotent Lie algebras. Nevertheless, a precise description of their symmetries, as given by the understanding of their automorphism group, is yet to be obtained (see [5, 6] and [7, 1]). The problem of describing the automorphism group seems to be considerably simpler when a deformation is introduced.

It is reasonable to think of a Noetherian generalized down-up algebra as a deformation of an enveloping algebra of a 3-dimensional Lie algebra. Working over an algebraically closed field of characteristic 0, we use elementary methods to compute the automorphism groups of Noetherian generalized down-up algebras, under certain assumptions. Specializing, our results to down-up algebras, we obtain a complete description of the automorphism groups of all Noetherian down-up algebras $A(\alpha, \beta, \gamma)$, under the restriction that at least one of the roots of the polynomial $X^2 - \alpha X - \beta$ is not a root of unity.

1 The main results

$\mathbb{K}$ denotes an algebraically closed field of characteristic 0 and $\mathbb{K}^*$ is the multiplicative group of units of $\mathbb{K}$.

Definition 1. *Let $f \in \mathbb{K}[X]$ be a polynomial and fix scalars $r, s, \gamma \in \mathbb{K}$. The generalized down-up algebra $L = L(f, r, s, \gamma)$ is the unital associative $\mathbb{K}$-algebra generated by d, u and h, subject to the relations:*

$$dh - rhd + \gamma d = 0, \quad hu - ruh + \gamma u = 0, \quad du - sud + f(h) = 0.$$

L is said to be conformal if there exists a polynomial $g \in \mathbb{K}[X]$ such that $f(X) = sg(X) - g(rX - \gamma)$.

We denote the group of algebra automorphisms of L by Aut. Consider the following subgroup of Aut: $\mathcal{H} = \{\phi \in \mathrm{Aut} \mid \phi(h) = \lambda h, \text{ for some } \lambda \in \mathbb{K}^*\}$.

Theorem 1. *Let $L = L(f, r, s, \gamma)$ be a generalized down-up algebra. Assume r, $s \in \mathbb{K}^*$, r is not a root of unity and $f(X) = sg(X) - g(rX - \gamma)$ for some $g \in \mathbb{K}[X]$. Then the group* Aut *of algebra automorphisms of L is isomorphic to:*

1. *$(\mathbb{K}^*)^3 \rtimes \mathbb{Z}/3\mathbb{Z}$ if $f = 0$ and $s = r^{-1}$, where the generator $1 + 3\mathbb{Z}$ of $\mathbb{Z}/3\mathbb{Z}$ acts on the torus $(\mathbb{K}^*)^3$ via the automorphism $(\lambda_1, \lambda_2, \lambda_3) \mapsto (\lambda_3, \lambda_1, \lambda_2)$;*
2. *$\mathbb{K} \rtimes (\mathbb{K}^*)^3$ if $f = 0$ and $s^\tau = r$, where $(\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{K}^*)^3$ acts on the additive group $\mathbb{K}$ via the automorphism $t \mapsto \lambda_1^{-1} (\lambda_2 \lambda_3)^\tau t$;*
3. *$\mathbb{K} \rtimes (\mathbb{K}^*)^2$ if $\deg(f) = 0$ and $s^\tau = r$, where $(\lambda_1, \lambda_2) \in (\mathbb{K}^*)^2$ acts on the additive group $\mathbb{K}$ via the automorphism $t \mapsto \lambda_1^{-1} t$;*

4. $(\mathbb{K}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ *if* $\deg(f) = 1$, $s = r^{-1}$ *and* $f(\frac{\gamma}{r-1}) = 0$, *where the generator* $1+2\mathbb{Z}$ *of* $\mathbb{Z}/2\mathbb{Z}$ *acts on the torus* $(\mathbb{K}^*)^2$ *via the automorphism* $(\lambda_1, \lambda_2) \mapsto (\lambda_1, \lambda_2^{-1}\lambda_1)$*;*

5. $\mathbb{K}^* \rtimes \mathbb{Z}/2\mathbb{Z}$ *if* $\deg(f) = 1$, $s = r^{-1}$ *and* $f(\frac{\gamma}{r-1}) \neq 0$, *where the generator* $1 + 2\mathbb{Z}$ *of* $\mathbb{Z}/2\mathbb{Z}$ *acts on the torus* $\mathbb{K}^*$ *via the automorphism* $\lambda \mapsto \lambda^{-1}$*;*

6. $\mathcal{H}$ *otherwise, where* $\mathcal{H}$ *can be explicitly described.*

Other classes of algebras to which our study applies are Le Bruyn's *conformal* $\mathfrak{sl}_2$ *enveloping algebras* [8], occuring as $L(bx^2+x, r, s, \gamma)$, for $b \in \mathbb{K}$ and $rs \neq 0$, and some of Witten's seven parameter deformations of the enveloping algebra of $\mathfrak{sl}_2$ [11] (see also [3, Thm. 2.6] and [4, Ex. 1.4]).

For the convenience of those readers who are mostly interested in down-up algebras, we replace our usual generators x and y of L with the canonical generators d and u of A, respectively.

Theorem 2. *Let* $A = A(\alpha, \beta, \gamma)$ *be a down-up algebra, with* $\alpha = r + s$ *and* $\beta = -rs$. *Assume that* $\beta \neq 0$ *and that one of* r *or* s *is not a root of unity. The group* $\mathrm{Aut(A)}$ *of algebra automorphisms of* A *is described bellow.*

1. *If* $\gamma = 0$ *and* $\beta = -1$ *then* $\mathrm{aut(A)} \simeq (\mathbb{K}^*)^2 \rtimes \mathbb{Z}/2\mathbb{Z}$*;*

2. *If* $\gamma = 0$ *and* $\beta \neq -1$ *then* $\mathrm{Aut(A)} \simeq (\mathbb{K}^*)^2$*;*

3. *If* $\gamma \neq 0$ *and* $\beta = -1$ *then* $\mathrm{aut(A)} \simeq \mathbb{K}^* \rtimes \mathbb{Z}/2\mathbb{Z}$*;*

4. *If* $\gamma \neq 0$ *and* $\beta \neq -1$ *then* $\mathrm{Aut(A)} \simeq \mathbb{K}^*$.

In all cases, the 2*-torus* $(\mathbb{K}^*)^2$ *acts diagonally on the generators* d *and* u, $\mu \in \mathbb{K}^*$ *acts as multiplication by* μ *on* d *and as multiplication by* μ^{-1} *on* u, *and the generator of the finite group* $\mathbb{Z}/2\mathbb{Z}$ *interchanges* d *and* u.

References

[1] J. Alev, *Un automorphisme non modéré de* $U(\mathfrak{g}_3)$, Comm. Algebra **14** (1986), no. 8, 1365–1378.

[2] G. Benkart and T. Roby, *Down-up algebras*, J. Algebra **209** (1998), no. 1, 305–344.

[3] G. Benkart, *Down-up algebras and Witten's deformations of the universal enveloping algebra of* $\mathfrak{s}l_2$, Contemp. Math. **224** (1999), 29–45.

[4] T. Cassidy and B. Shelton, *Basic properties of generalized down-up algebras*, J. Algebra **279** (2004), no. 1, 402–421.

[5] J. Dixmier, *Sur les algèbres de Weyl*, Bull. Soc. Math. France **96** (1968), 209–242.

[6] J. Dixmier, *Quotients simples de l'algèbre enveloppante de* $\mathfrak{s}l_2$, J. Algebra **24** (1973), 551–564.

[7] A. Joseph, *A wild automorphism of* Usl(2), Math. Proc. Cambridge Philos. Soc. **80** (1976), no. 1, 61–65.

[8] L. Le Bruyn, *Conformal* sl_2 *enveloping algebras*, Commun. Algebra **23** (1995), no. 4, 1325–1362.

[9] S. Rueda, *Some algebras similar to the enveloping algebra of* $sl(2)$, Commun. Algebra **30** (2002), no. 3, 1127–1152.

[10] S.P. Smith, *A class of algebras similar to the enveloping algebra of sl(2)*, Trans. Am. Math. Soc. **322** (1990), no. 1, 285–314.

[11] E. Witten, *Gauge theories, vertex models, and quantum groups*, Nuclear Phys. B **330** (1990), 285–346.