Homotopy pullbacks for n-groupoids

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Abstract

In this talk, based on [5], I will have a glance to a convenient notion of pullback for n-categories and n-groupoids. The main application is the study of exact sequences for pointed n-groupoids.

The starting point is a result due to R. Brown [1]: given a functor $A \to B$ between pointed groupoids and its homotopy fibre $\mathcal{F}_*$, there is a 6-term exact sequence of pointed sets

$$\mathcal{F}_* \to \mathcal{A} \to \mathcal{B}$$

$$\pi_1\mathcal{F}_* \to \pi_1 A \to \pi_1 B \to \pi_0\mathcal{F}_* \to \pi_0 A \to \pi_0 B$$

More recently, starting from a morphism $A \to B$ of pointed 2-groupoids and using convenient notions of homotopy fibre $\mathcal{F}_*$ and exactness, a 6-term exact sequence of pointed groupoids and a 9-term exact sequence of pointed sets have been obtained in [2] and [3]

$$\mathcal{F}_* \to A \to B$$

$$\pi_1\mathcal{F}_* \to \pi_1 A \to \pi_1 B \to \pi_0\mathcal{F}_* \to \pi_0 A \to \pi_0 B$$

$$\pi_1\pi_1\mathcal{F}_* \to \pi_1\pi_1 A \to \pi_1\pi_1 B \to \pi_1\pi_0\mathcal{F}_* \to \pi_1\pi_0 A \to \pi_1\pi_0 B \to \pi_0\pi_0\mathcal{F}_* \to \pi_0\pi_0 A \to \pi_0\pi_0 B$$

It is therefore quite natural to ask if, given a morphism $A \to B$ of pointed n-groupoids, is it possible to construct an $(n+1)$-level "ziqqurath" of exact sequences

a 3-term exact sequence of pointed n-groupoids

a 6-term exact sequence of pointed $(n-1)$-groupoids a 9-term exact sequence of pointed $(n-2)$-groupoids

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a $3n$-term exact sequence of pointed groupoids

a $3(n+1)$-term exact sequence of pointed sets

Clearly, to make precise such a question we have to understand what is an homotopy fibre and more in general an homotopy pullback in the context of n-groupoids and what it means exactness for a sequence of pointed n-groupoids.

The main steps to work out in details the previous program are:
1) An inductive definition of the sesqui-category $nCat$ of n-categories, n-functors and lax n-transformations.
2) An inductive description of homotopy pullbacks in $nCat$.
3) An inductive definition of n-groupoids using equivalences of $(n-1)$-groupoids. The definition we give is equivalent to those given by Kapranov-Voevodsky [4] and by Street [6].
4) The construction of sesqui-functors $\pi_0,\ \pi_1 \colon nGrpd_* \to (n-1)Grpd_*$. The construction of π_1 is the most delicate point: one needs to consider n-modifications and to show that homotopy pullbacks in $nCat$ satisfy a universal property involving n-modifications.
5) A definition of exactness for a sequence of pointed n-groupoids, using homotopy fibers.

Once these definitions and constructions achieved, we get the desired results:
1) If $A \to B \to C$ is an exact sequence of pointed n-groupoids, then $\pi_0 A \to \pi_0 B \to \pi_0 C$ and $\pi_1 A \to \pi_1 B \to \pi_1 C$ are exact sequences of pointed $(n-1)$-groupoids.
2) Given an n-functor $A \to B$ between pointed n-groupoids together with its homotopy fibre $\mathcal{F}_*$, it is possible to construct a 6-term exact sequence of pointed $(n-1)$-groupoids $\pi_1\mathcal{F}_* \to \pi_1 A \to \pi_1 B \to \pi_0 \mathcal{F}_* \to \pi_0 A \to \pi_0 B$
3) Since π_0 and π_1 commute, $\pi_0 \cdot \pi_1 \simeq \pi_1 \cdot \pi_0$, applying π_0 and π_1 to the exact sequerce of the previous point one gets a 9-term exact sequence of pointed $(n-2)$-groupoids. By iteration, we have a ziqqurath of exact sequences of increasing length (from the top to the bottom) and decreasing dimension.

References

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